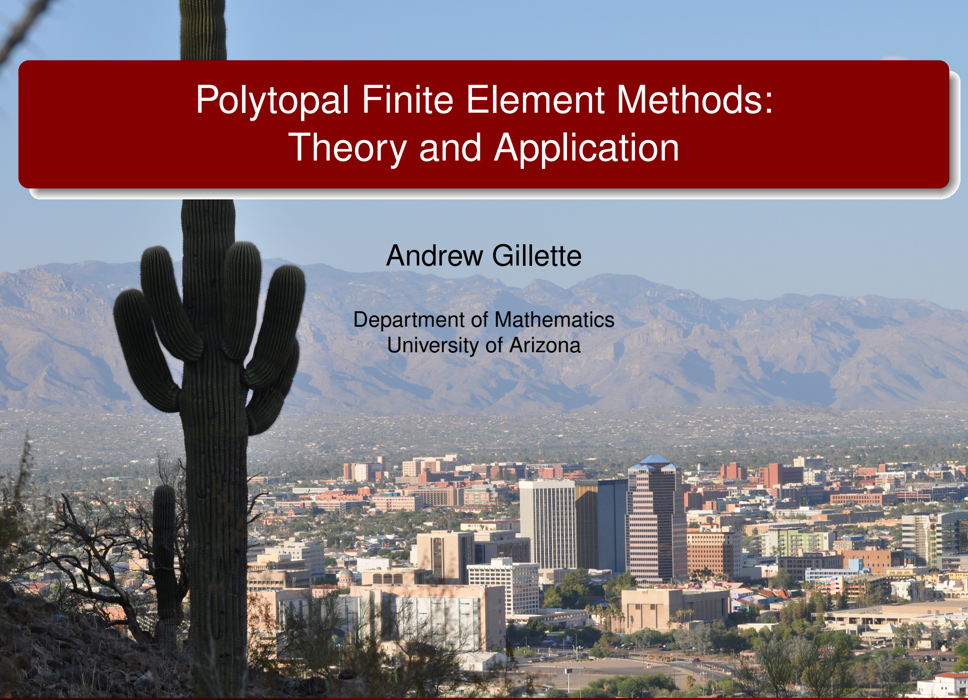


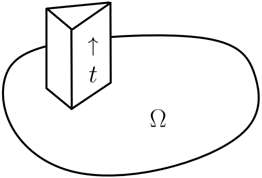
Polytopal Finite Element Methods: Theory and Application

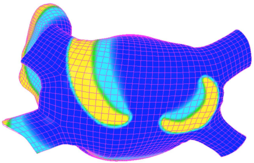
Andrew Gillette

Department of Mathematics
University of Arizona

What are finite element methods?

The **finite element method** is a way to numerically approximate the solution to PDEs.





$$\begin{bmatrix} \mathbb{A} \end{bmatrix} \begin{bmatrix} \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \end{bmatrix}$$

CHARACTERIZE	DISCRETIZE	SOLVE
Real analysis	Geometry & Topology	Linear algebra
PDEs	Combinatorics	Numerical analysis

In this talk:

- Why use meshes of polygons / polyhedra for discretization?
- What are some finite element methods that allow such meshes?
- Where is this line of research headed?

Table of Contents

Outline

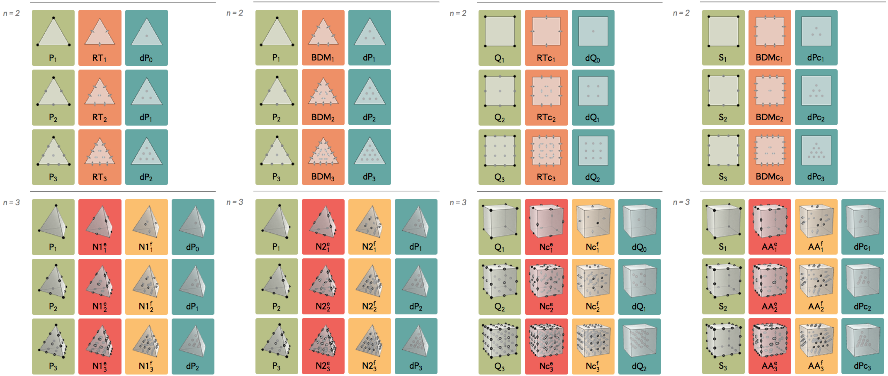
1. Why use polytopal meshes?

2. What are some polyhedral finite element methods?

3. Where is this line of research headed?

Discretization with Simplices or Cubes

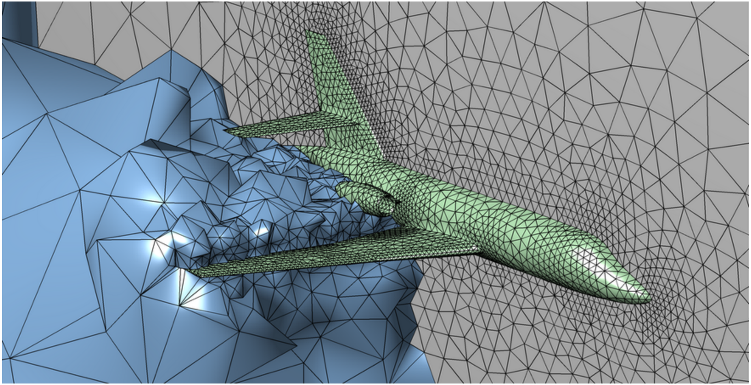
Domain meshing with simplices or cubes is now so well-understood that there is a **Periodic Table of Finite Elements:**



→ Viewable online at `femtable.org`

→ Scientific content prepared by Doug Arnold and Anders Logg

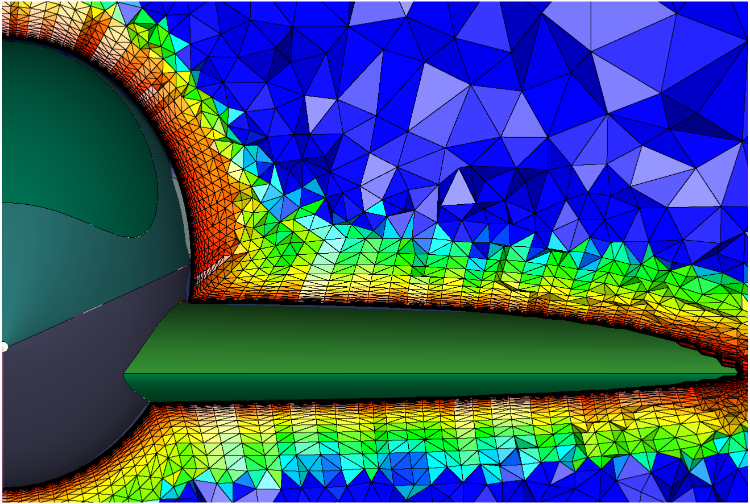
Volume meshing for Computational Fluid Dynamics



*Tetrahedral volume mesh for CFD, using **DistMesh** software.*

(courtesy of Per-Olof Persson)

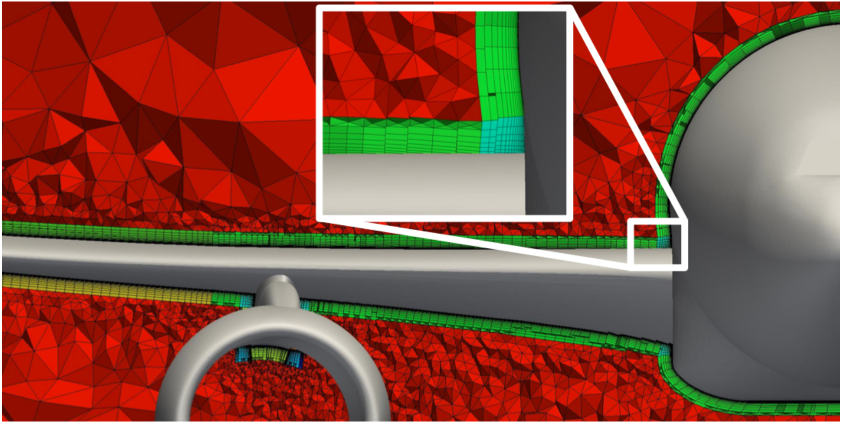
Volume meshing for Computational Fluid Dynamics



*Tetrahedral volume mesh for CFD, using **Pointwise** software.*

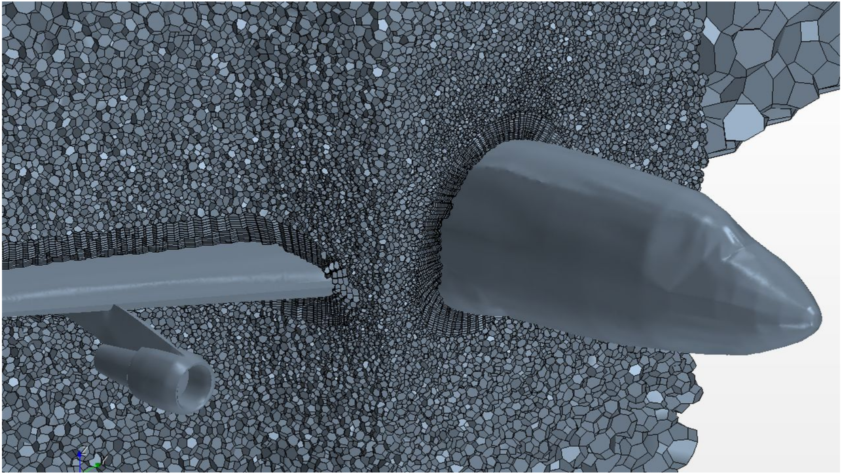
(from blog.pointwise.com)

Volume meshing for Computational Fluid Dynamics



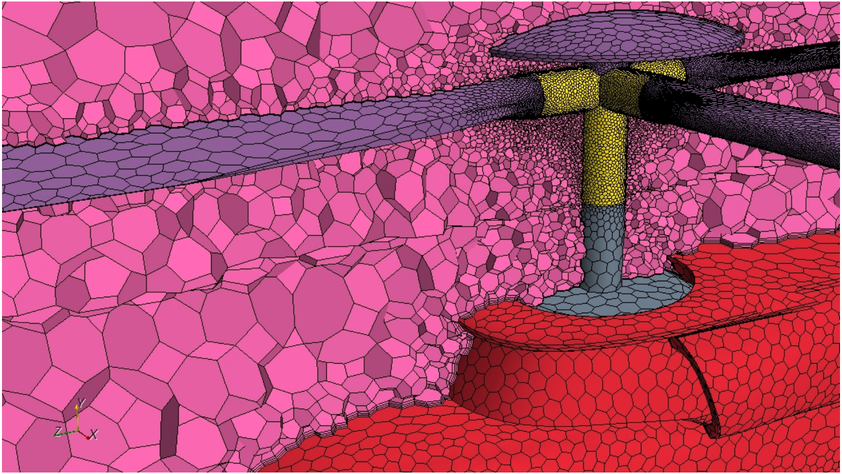
*Hyrbid hex / pyramid / prism / tet mesh for CFD, using **ITI Transcendata** software.*
(from a keynote address at Geometric Modeling and Processing 2015)

Volume meshing for Computational Fluid Dynamics



*Body-aligned prismatic polyhedral meshes for CFD, using **CD-adapco** software.*
(from cd-adapco.com image gallery)

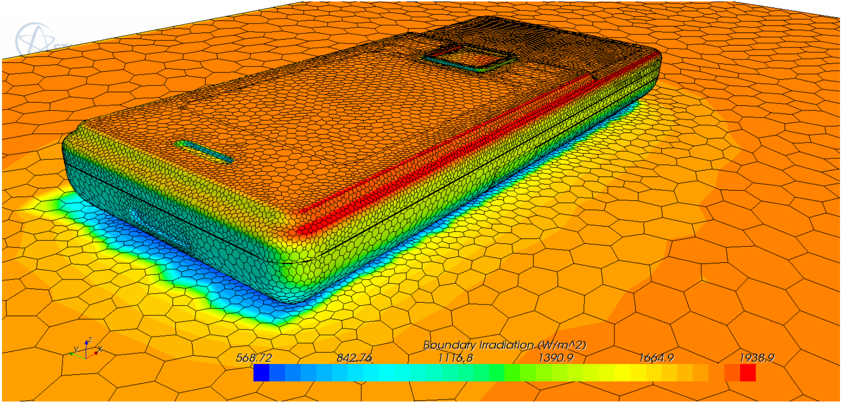
Volume meshing for Computational Fluid Dynamics



Polyhedral mesh of a Bell 407 helicopter and surrounding volume.

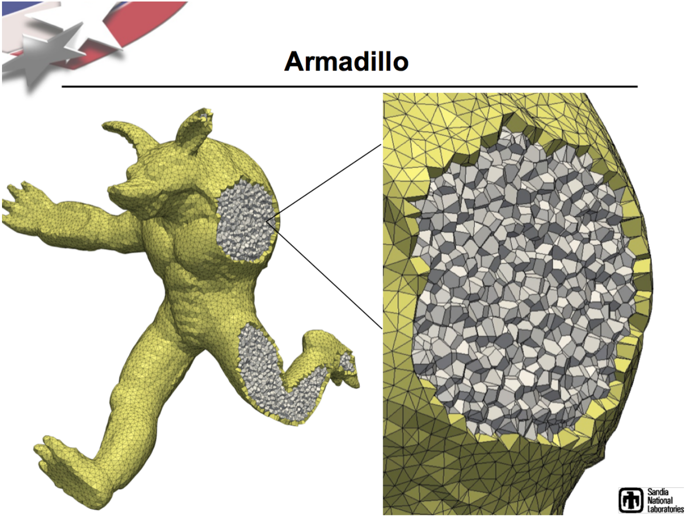
(from cd-adapco.com image gallery)

Volume meshing for. . . cel phone design!



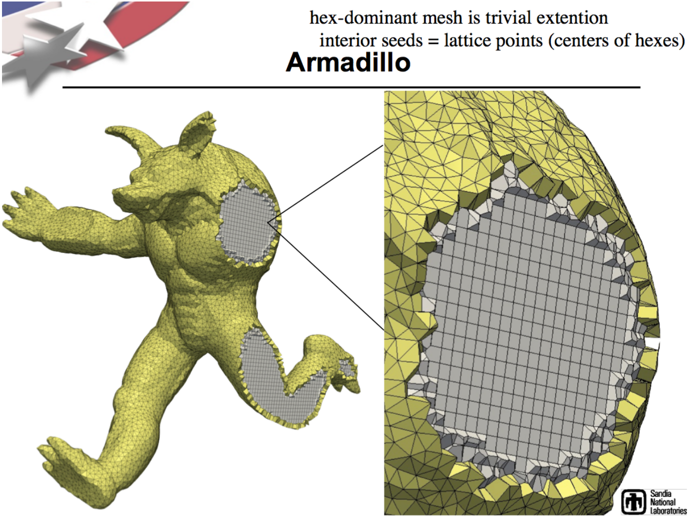
A polyhedral mesh used to study heat transfer and cooling of a cell phone.
(from cd-adapco.com image gallery)

Volume meshing at Sandia National Labs



A polyhedral mesh conforming to a surface triangulation using ***VoroCrust*** *software.*
(from Scott Mitchell, Sandia National Labs)

Volume meshing at Sandia National Labs



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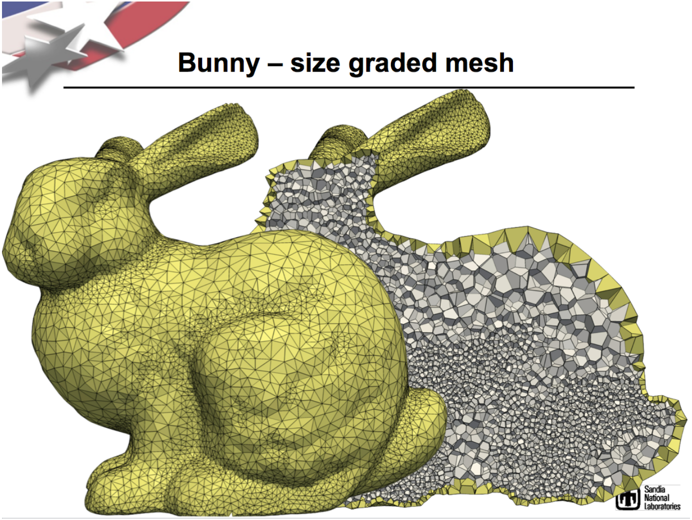
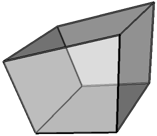
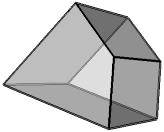


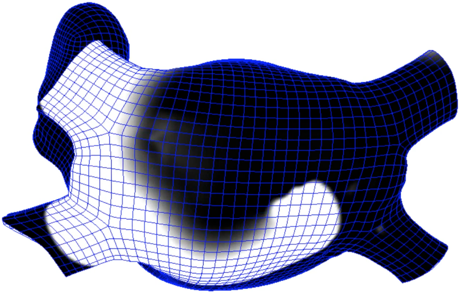
*The requisite Stanford Bunny example using **VoroCrust** software.*
(from Scott Mitchell, Sandia National Labs)

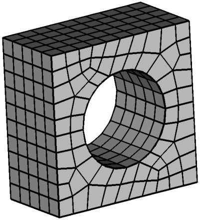
Hexahedral meshing is polyhedral meshing

Meshes of generic hexahedra require a generalized theory of polyhedral discretization, related to but distinct from the theory for perfect tensor product meshes.







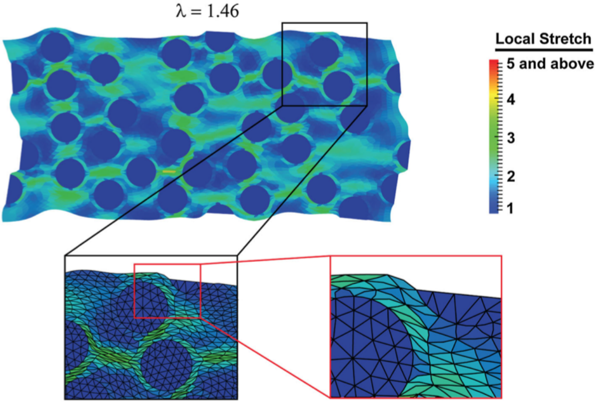


↑ *Heart mesh made using Continuity software, National Biomedical Computation Resource, UCSD*

← *Hole mesh made using CUBIT Geometry and Mesh Generation Toolkit, Sandia National Labs*

Elasticity modeling

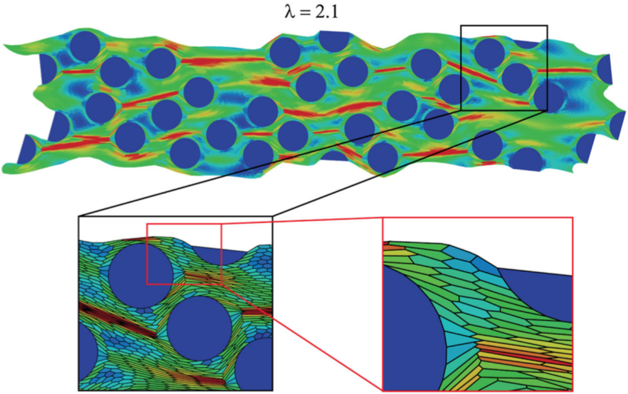
Standard triangular FEM cannot model maximal stretch factors due to numerical errors from the deformation.



(from Chi, Talischi, Lopez-Pamies, Paulino, “Polygonal finite elements for finite elasticity.” International Journal for Numerical Methods in Engineering, 2015)

Elasticity modeling

The flexibility of polyhedral meshes allows greater shape deformation and more realistic stretch factors.



Chi et al. "Polygonal finite elements for finite elasticity."
Talischi et al. "Gradient correction for polygonal and polyhedral finite elements."
International Journal for Numerical Methods in Engineering, 2015

Outline

1. Why use polytopal meshes?
2. What are some polyhedral finite element methods?
3. Where is this line of research headed?

The Finite Element Method for Poisson's Problem

Continuous problem: find $u \in U$ s.t.

$$\Delta u = f \text{ on } \Omega \subset \mathbb{R}^n$$

Weak form: find $u \in U$ s.t.

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f\ v, \quad \forall v \in V \quad (\dim V = \infty)$$

Discrete form: find $u_h \in U_h$ s.t.

$$\int_\Omega \nabla u_h \cdot \nabla v_h = \int_\Omega f\ v_h, \quad \forall v_h \in V_h \quad (\dim V_h < \infty)$$

Linear system: Set $U_h := V_h$ (Galerkin method). Find $u \in \mathbb{R}^{\dim V_h}$ solving

$$[\,\mathbb{K}\,]_{ji}\ [\,u\,]_i = [\,f\,]_j\,, \quad \forall v_j \in \text{basis for } V_h$$

where

$$[\,\mathbb{K}\,]_{ji} = \int_E \nabla v_i \cdot \nabla v_j \quad \text{and} \quad [\,f\,]_j = \int_E f(x) v_j(x)$$

Key challenge when E is a polygon/polyhedron:

Efficient computation or approximation of $\int_E \nabla v_i \cdot \nabla v_j, \quad \forall v_j \in \text{basis for } V_h$

A few kinds of polytopal element methods. . .

CDO = Compatible discrete operator schemes

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"Polytopal Element Methods in Mathematics and Engineering"

→ Special NSF-funded workshop held at Georgia Tech in Oct 2015
→ Slides from talks: `http://www.poems15.gatech.edu/`

Outline

Future directions for polytopal element methods

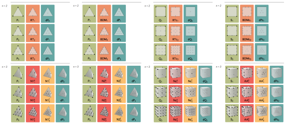
- Robust software for polygonal meshing is freely available. . .

  but robust **polyhedral meshing** is still in development.
- Industry and engineering researchers are interested in polytopal methods. . .

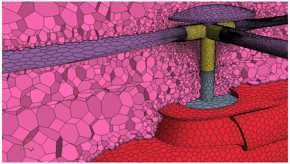
  but **increased communication** will be crucial for widespread adoption.
- There is no ‘silver bullet’ method. . .

  so there are **lots of open questions**!

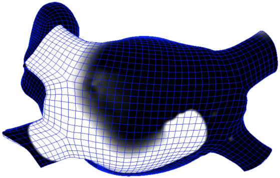
My main interests are:



finite element theory



polyhedral elements



biomedical applications

Acknowledgments

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Collaborators

Chandrajit Bajaj	UT Austin	computer science
Snorre Christiansen	U. Oslo	math
Michael Floater	U. Oslo	math
Michael Holst	UC San Diego	math
Peter Kekenes-Huskey	U. Kentucky	chemistry
Alexander Rand	CD-adapco	industry
N. Sukumar	UC Davis	civil engineering
Kevin Vincent	UC San Diego	bioengineering
Yunrong Zhu	Idaho State	math

Slides and pre-prints: `http://math.arizona.edu/~agillette/`