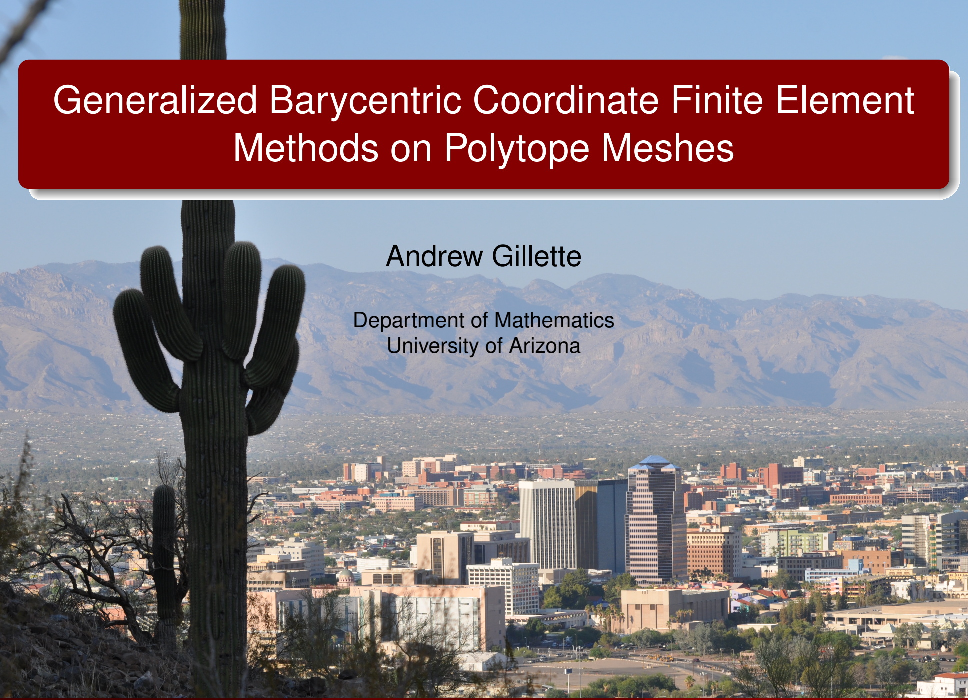

Generalized Barycentric Coordinate Finite Element Methods on Polytope Meshes

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What are *a priori* FEM error estimates?

Poisson's equation in $\mathbb{R}^n$: Given a domain $\mathcal{D} \subset \mathbb{R}^n$ and $f : \mathcal{D} \to \mathbb{R}$, find u such that

$$\begin{array}{rrcll} \text{strong form} & -\Delta u & = & f & u \in H^2(\mathcal{D}) \\ \text{weak form} & \displaystyle\int_{\mathcal{D}} \nabla u \cdot \nabla \phi & = & \displaystyle\int_{\mathcal{D}} f\,\phi & \forall \phi \in H^1(\mathcal{D}) \\ \text{discrete form} & \displaystyle\int_{\mathcal{D}} \nabla u_h \cdot \nabla \phi_h & = & \displaystyle\int_{\mathcal{D}} f\,\phi_h & \forall \phi_h \in V_h \leftarrow \textit{finite dim.} \subset H^1(\mathcal{D}) \end{array}$$

Typical **finite element method:**

→ Mesh $\mathcal{D}$ by polytopes $\{P\}$ with vertices $\{\mathbf{v}_i\}$; define $h := \max \operatorname{diam}(P)$.

→ Fix $\boxed{\text{basis functions } \lambda_i}$ with local piecewise support, e.g. barycentric functions.

→ Define u_h such that it uses the λ_i to approximate u, e.g. $\boxed{u_h := \sum_i u(\mathbf{v}_i)\lambda_i}$

A linear system for u_h can then be derived, admitting an ***a priori* error estimate:**

$$\underbrace{||u - u_h||_{H^1(P)}}_{\text{approximation error}} \leq \underbrace{C\, h^{\mathbf{p}}\, |u|_{H^{\mathbf{p+1}}(P)}}_{\text{optimal error bound}}, \quad \forall u \in H^{\mathbf{p+1}}(P),$$

provided that the λ_i span all degree p polynomials on each polytope P.

The generalized barycentric coordinate approach

Let P be a convex polytope with vertex set V. We say that

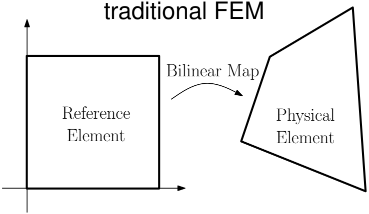
$$\lambda_{\mathbf{v}} : P \to \mathbb{R} \quad \text{are} \quad \textbf{generalized barycentric coordinates (GBCs)} \text{ on } P$$

if they satisfy $\quad \lambda_{\mathbf{v}} \geq 0$ on P and $\displaystyle L = \sum_{\mathbf{v}\in V} L(\mathbf{v_v})\lambda_{\mathbf{v}}, \quad \forall\, L : P \to \mathbb{R}$ linear.

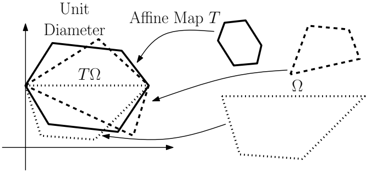
Familiar properties are implied by this definition:

$$\underbrace{\sum_{\mathbf{v}\in V} \lambda_{\mathbf{v}} \equiv 1}_{\text{partition of unity}} \qquad \underbrace{\sum_{\mathbf{v}\in V} \mathbf{v}\lambda_{\mathbf{v}}(\mathbf{x}) = \mathbf{x}}_{\text{linear precision}} \qquad \underbrace{\lambda_{\mathbf{v}_i}(\mathbf{v}_j) = \delta_{ij}}_{\text{interpolation}}$$

traditional FEM

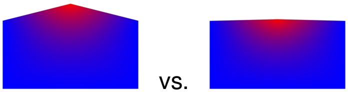


family of **GBC** reference elements

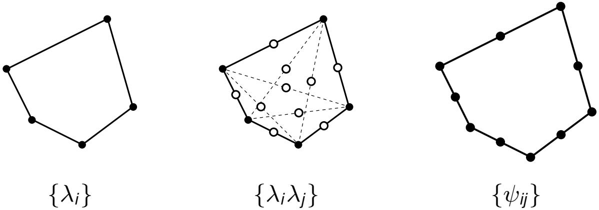


Developments in GBC FEM theory

1 Characterization of the dependence of error estimates on polytope geometry.



2 Construction of higher order scalar-valued methods using $\lambda_{\mathbf{v}}$ functions.



3 Construction of $H(\text{curl})$ and $H(\text{div})$ methods using $\lambda_{\mathbf{v}}$ and $\nabla\lambda_{\mathbf{v}}$ functions.

$$\begin{tikzcd}
H^1 \arrow[r, "\text{grad}"] & H(\text{curl}) \arrow[r, "\text{curl}"] & H(\text{div}) \arrow[r, "\text{div}"] & L^2 \\
\{\lambda_i\} & \{\lambda_i\nabla\lambda_j\} & \{\lambda_i\nabla\lambda_j \times \nabla\lambda_k\} & \{\chi_P\}
\end{tikzcd}$$

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2. Quadratic serendipity elements on polygons

3. Basis construction for vector-valued problems

4. Numerical results

Anatomy of an error estimate

In the case of functions $\lambda_{\mathbf{v}}$ associated to vertices of a *polygonal* mesh, we have:

$$\underbrace{\left\| u - \sum_{\mathbf{v}} u(\mathbf{v})\lambda_{\mathbf{v}} \right\|_{H^1(P)}}_{\substack{\text{approximation error} \\ \text{in value and derivative}}} \leq \underbrace{C_{\textit{proj-}\lambda} \quad C_{\textit{proj-linear}} \quad \operatorname{diam}(P)}_{\text{constants}} \quad \underbrace{|u|_{H^2(P)}}_{\substack{\text{2nd order} \\ \text{oscillation} \\ \text{in } u}}$$

$C_{\textit{proj-}\lambda}$ $\approx$ operator norm of projection $u \longmapsto \sum_{\mathbf{v}} u(\mathbf{v})\lambda_{\mathbf{v}}$

$C_{\textit{proj-linear}}$ $\approx$ operator norm of projection $u \longmapsto$ linear polynomials on P (from Bramble-Hilbert Lemma)

$\operatorname{diam}(P)$ $=$ diameter of polygon P.

Key question for polygonal finite element methods

What geometrical properties of P can cause $C_{\textit{proj-}\lambda}$ to be large?

Mathematical characterization

Problem statement

Given a simple convex d-dimensional polytope P, define

$$\Lambda := \sup_{\mathbf{x} \in P} \sum_{\mathbf{v} \in V} |\nabla \lambda_{\mathbf{v}}(\mathbf{x})|$$

where $\lambda_{\mathbf{v}}$ are **generalized barycentric coordinates** on P.

Find upper and lower bounds on Λ in terms of geometrical properties of P.

Remark: It can be shown that

$$C_{\textit{proj-}\lambda} = 1 + C_S(1 + \Lambda)$$

where C_S is the Sobolev embedding constant satisfying $||u||_{C^0(\overline{P})} \leq C_S \, ||u||_{H^k(P)}$ independent of $u \in H^k(P)$, provided that $k > d/2$.

Hence, bounds on Λ help us characterize when $C_{\textit{proj-}\lambda}$ is large.

The triangular case

$$\Lambda := \sup_{\mathbf{x}\in P} \sum_{\mathbf{v}\in V} |\nabla \lambda_{\mathbf{v}}(\mathbf{x})|$$

If P is a triangle, Λ can be large when P has a large interior angle.

→ This is often called the **maximum angle condition** for finite elements.

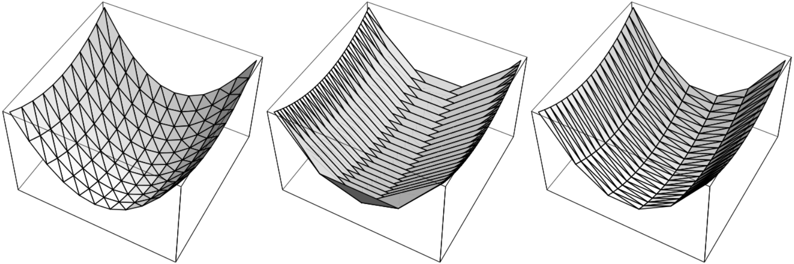
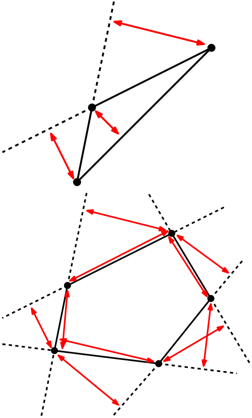


Figure from: SHEWCHUK *What is a good linear element?* Int'l Meshing Roundtable, 2002.

BABUŠKA, AZIZ *On the angle condition in the finite element method*, SIAM J. Num. An., 1976.

JAMET *Estimations d'erreur pour des éléments finis droits presque dégénérés*, ESAIM:M2AN, 1976.

Motivation



Observe that on triangles of fixed diameter:

$$|\nabla\lambda_{\mathbf{v}}| \text{ large } \iff \text{ interior angle at } \mathbf{v} \text{ is large}$$
$$\iff \text{ the altitude "at } \mathbf{v}\text{" is small}$$

For Wachspress coordinates, we generalize to polygons:

$$|\nabla\lambda_{\mathbf{v}}| \text{ large } \iff \text{ the "altitude" at } \mathbf{v} \text{ is small}$$

and then to **simple** polytopes.

(A simple *d*-dimensional polytope
has exactly *d* faces at each vertex)

Given a simple convex *d*-dimensional polytope *P*, let

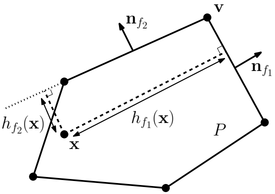
$$h_* := \text{ minimum distance from a vertex to a hyper-plane of a non-incident face.}$$

$$\text{Then} \qquad \sup_{\mathbf{x}\in P} \sum_{\mathbf{v}\in V} |\nabla\lambda_{\mathbf{v}}(\mathbf{x})| =: \Lambda \text{ is large} \iff h_* \text{ is small}$$

Upper bound for simple convex polytopes

Theorem [Floater, G., Sukumar]

Let P be a simple convex polytope in $\mathbb{R}^d$ and let $\lambda_{\mathbf{v}}$ be **generalized Wachspress coordinates**. Then $\Lambda \leq \dfrac{2d}{h_*}$ where $h_* = \min\limits_{f} \min\limits_{\mathbf{v} \notin f} \operatorname{dist}(\mathbf{v}, f)$



$$\mathbf{p}_f(\mathbf{x}) := \frac{\mathbf{n}_f}{h_f(\mathbf{x})} =$$ normal to face f, scaled by the reciprocal of the distance from $\mathbf{x}$ to f

$$w_{\mathbf{v}}(\mathbf{x}) := \det(\mathbf{p}_{f_1}(\mathbf{x}), \cdots, \mathbf{p}_{f_d}(\mathbf{x}))$$

= volume formed by the d vectors $\{\mathbf{p}_{f_i}(\mathbf{x})\}$ for the d faces incident to $\mathbf{v}$

The **generalized Wachspress coordinates** are defined by

$$\lambda_{\mathbf{v}}(\mathbf{x}) := \frac{w_{\mathbf{v}}(\mathbf{x})}{\sum\limits_{\mathbf{u}} w_{\mathbf{u}}(\mathbf{x})}$$

Proof sketch for upper bound

To prove: $\sup_{\mathbf{x}} \sum_{\mathbf{v}} |\nabla \lambda_{\mathbf{v}}(\mathbf{x})| =: \Lambda \leq \frac{2d}{h_*}$ where $h_* := \min_f \min_{\mathbf{v} \notin f} h_f(\mathbf{v})$.

1. Bound $|\nabla \lambda_{\mathbf{v}}|$ by summations over faces incident and not incident to **v**.

$$|\nabla \lambda_{\mathbf{v}}| \leq \lambda_{\mathbf{v}} \sum_{f \in F_{\mathbf{v}}} \frac{1}{h_f} \left(1 - \sum_{\mathbf{u} \in f} \lambda_{\mathbf{u}}\right) + \lambda_{\mathbf{v}} \sum_{f \notin F_{\mathbf{v}}} \frac{1}{h_f} \left(\sum_{\mathbf{u} \in f} \lambda_{\mathbf{u}}\right)$$

2. Summing over **v** gives a constant bound.

$$\sum_{\mathbf{v}} |\nabla \lambda_{\mathbf{v}}| \leq 2 \sum_{f \in F} \frac{1}{h_f} \left(1 - \sum_{\mathbf{u} \in f} \lambda_{\mathbf{u}}\right) \left(\sum_{\mathbf{u} \in f} \lambda_{\mathbf{u}}\right)$$

3. Write $h_f(\mathbf{x})$ using $\lambda_{\mathbf{v}}$ (possible since h_f is linear) and derive the bound.

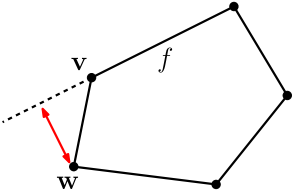
$$\Lambda \leq 2 \sum_{f \in F} \left(\sum_{\mathbf{u} \in f} \lambda_{\mathbf{u}}\right) \frac{1}{h_*} = 2 \sum_{\mathbf{v} \in V} |\{f \ : \ f \ni \mathbf{v}\}| \lambda_{\mathbf{v}} \frac{1}{h_*} = \frac{2d}{h_*}$$

Lower bound for polytopes

Theorem [Floater, G., Sukumar]

Let P be a simple convex polytope in $\mathbb{R}^d$ and let $\lambda_{\mathbf{v}}$ be **any** generalized barycentric coordinates on P. Then

$$\frac{1}{h_*} \leq \Lambda$$



Proof sketch:

1. Show that $h_* = h_f(\mathbf{w})$, for some particular face f of P and vertex $\mathbf{w} \notin f$.
2. Let $\mathbf{v}$ be the vertex in f closest to $\mathbf{w}$. Show that
$$|\nabla \lambda_{\mathbf{w}}(\mathbf{v})| = \frac{1}{h_f(\mathbf{w})}$$
3. Conclude the result, since
$$\Lambda \geq |\nabla \lambda_{\mathbf{w}}(\mathbf{v})| = \frac{1}{h_f(\mathbf{w})} = \frac{1}{h_*}$$

Upper and lower bounds on polytopes

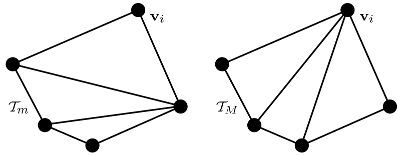
For a polytope $P \subset \mathbb{R}^d$, define $\Lambda := \sup_{\mathbf{x} \in P} \sum_{\mathbf{v}} |\nabla \lambda_{\mathbf{v}}(\mathbf{x})|$.

simple convex polytope in $\mathbb{R}^d$	$\frac{1}{h_*}$	$\leq$	Λ	$\leq$	$\frac{2d}{h_*}$
d-simplex in $\mathbb{R}^d$	$\frac{1}{h_*}$	$\leq$	Λ	$\leq$	$\frac{d+1}{h_*}$
hyper-rectangle in $\mathbb{R}^d$	$\frac{1}{h_*}$	$\leq$	Λ	$\leq$	$\frac{d+\sqrt{d}}{h_*}$
regular k-gon in $\mathbb{R}^2$	$\frac{2(1+\cos(\pi/k))}{h_*}$	$\leq$	Λ	$\leq$	$\frac{4}{h_*}$

Note that $\lim_{k \to \infty} 2(1+\cos(\pi/k)) = 4$, so the bound is **sharp** in $\mathbb{R}^2$.

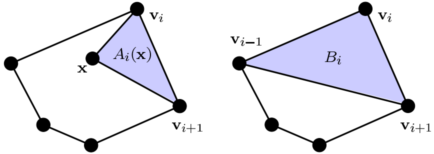
Floater, G, Sukumar *Gradient bounds for Wachspress coordinates on polytopes*, SIAM J. Numerical Analysis, 2014.

Many other barycentric coordinates are available . . .

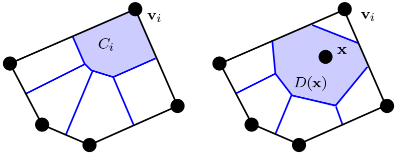


- Triangulation
  $\Rightarrow$ Floater, Hormann, Kós, *A general construction of barycentric coordinates over convex polygons*, 2006

$$0 \leq \lambda_i^{\mathcal{T}_m}(\mathbf{x}) \leq \lambda_i(\mathbf{x}) \leq \lambda_i^{\mathcal{T}_M}(\mathbf{x}) \leq 1$$

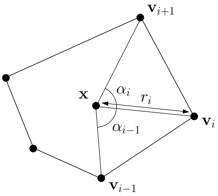


- Wachspress
  $\Rightarrow$ Wachspress, *A Rational Finite Element Basis*, 1975.
  $\Rightarrow$ Warren, *Barycentric coordinates for convex polytopes*, 1996.

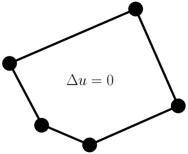


- Sibson / Laplace
  $\Rightarrow$ Sibson, *A vector identity for the Dirichlet tessellation*, 1980.
  $\Rightarrow$ Hiyoshi, Sugihara, *Voronoi-based interpolation with higher continuity*, 2000.

Many other barycentric coordinates are available . . .



- Mean value
  ⇒ FLOATER, *Mean value coordinates*, 2003.
  ⇒ FLOATER, KÓS, REIMERS, *Mean value coordinates in 3D*, 2005.



- Harmonic
  ⇒ WARREN, SCHAEFER, HIRANI, DESBRUN, *Barycentric coordinates for convex sets*, 2007.
  ⇒ CHRISTIANSEN, *A construction of spaces of compatible differential forms on cellular complexes*, 2008.

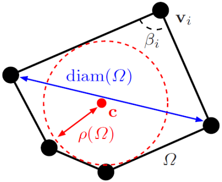
Many more papers could be cited (maximum entropy coordinates, moving least squares coordinates, surface barycentric coordinates, etc...)

Geometric criteria for convergence estimates

For other types of coordinates (on polygons only) we consider additional geometric measures.

Let $\rho(\Omega)$ denote the radius of the largest inscribed circle.
The **aspect ratio** γ is defined by

$$\gamma = \frac{\text{diam}(\Omega)}{\rho(\Omega)} \in (2, \infty)$$



Three possible geometric conditions on a polygonal mesh:

G1.	BOUNDED ASPECT RATIO:	$\exists\ \gamma^* < \infty$	such that	$\gamma < \gamma^*$
G2.	MINIMUM EDGE LENGTH:	$\exists\ d_* > 0$	such that	$\lvert\mathbf{v}_i - \mathbf{v}_{i-1}\rvert > d_*$
G3.	MAXIMUM INTERIOR ANGLE:	$\exists\ \beta^* < \pi$	such that	$\beta_i < \beta^*$

Polygonal Finite Element Optimal Convergence

Theorem [G, Rand, Bajaj]

In the table, any necessary geometric criteria to achieve the ***a priori* linear error estimate** are denoted by N. The set of geometric criteria denoted by S in each row ***taken together*** are sufficient to guarantee the estimate.

		G1 (aspect ratio)	**G2** (min edge length)	**G3** (max interior angle)
Triangulated	λ^{Tri}	-	-	S,N
Wachspress	λ^{Wach}	S	S	S,N
Sibson	λ^{Sibs}	S	S	-
Mean Value	λ^{MV}	S	S	-
Harmonic	λ^{Har}	S	-	-

G, RAND, BAJAJ *Error Estimates for Generalized Barycentric Interpolation*
Advances in Computational Mathematics, 37:3, 417-439, 2012

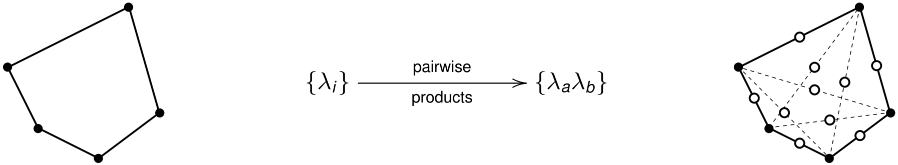
RAND, G, BAJAJ *Interpolation Error Estimates for Mean Value Coordinates*,
Advances in Computational Mathematics, 39:2, 327-347, 2013.

Outline

1. Error estimates for linear case
2. Quadratic serendipity elements on polygons
3. Basis construction for vector-valued problems
4. Numerical results

From linear to quadratic elements

A naïve quadratic element is formed by products of linear **GBCs**:



$$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\lambda_a\lambda_b\}$$

Why is this naïve?

- For a k-gon, this construction gives $k + \binom{k}{2}$ basis functions $\lambda_a\lambda_b$
- The space of quadratic polynomials is only dimension 6: $\{1, x, y, xy, x^2, y^2\}$
- Conforming to a linear function on the boundary requires 2 degrees of freedom per edge $\Rightarrow$ *only 2k functions needed!*

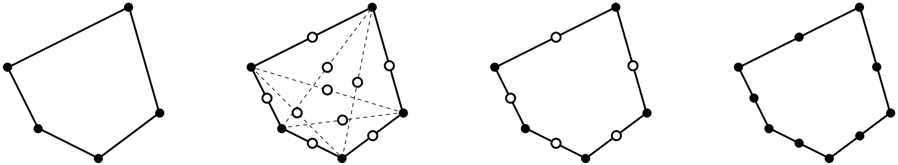
Problem Statement

Construct $2k$ basis functions associated to the vertices and edge midpoints of an arbitrary k-gon such that a quadratic convergence estimate is obtained.

Polygonal Quadratic Serendipity Elements

We define matrices $\mathbb{A}$ and $\mathbb{B}$ to reduce the naïve quadratic basis.

filled dot = **Interpolatory** domain point
= all functions in the set evaluate to 0
except the associated function which evaluates to 1
open dot = non-interpolatory domain point
= partition of unity satisfied, but not a nodal basis



$$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\lambda_a\lambda_b\} \xrightarrow{\mathbb{A}} \{\xi_{ij}\} \xrightarrow{\mathbb{B}} \{\psi_{ij}\}$$

k | $k + \binom{k}{2}$ | $2k$ | $2k$

Linear | Quadratic | Serendipity | Lagrange

From quadratic to serendipity

The bases are ordered as follows:

ξ_{ii} and $\lambda_a\lambda_a$ = basis functions associated with vertices
$\xi_{i(i+1)}$ and $\lambda_a\lambda_{a+1}$ = basis functions associated with edge midpoints
$\lambda_a\lambda_b$ = basis functions associated with interior diagonals,
i.e. $b \notin \{a-1, a, a+1\}$

Serendipity basis functions ξ_{ij} are a linear combination of pairwise products $\lambda_a\lambda_b$:

$$\begin{bmatrix} \xi_{ii} \\ \vdots \\ \xi_{i(i+1)} \end{bmatrix} = \mathbb{A} \begin{bmatrix} \lambda_a\lambda_a \\ \vdots \\ \lambda_a\lambda_{a+1} \\ \vdots \\ \lambda_a\lambda_b \end{bmatrix} = \begin{bmatrix} c_{11}^{11} & \cdots & c_{ab}^{11} & \cdots & c_{(n-2)n}^{11} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{11}^{ij} & \cdots & c_{ab}^{ij} & \cdots & c_{(n-2)n}^{ij} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{11}^{n(n+1)} & \cdots & c_{ab}^{n(n+1)} & \cdots & c_{(n-2)n}^{n(n+1)} \end{bmatrix} \begin{bmatrix} \lambda_a\lambda_a \\ \vdots \\ \lambda_a\lambda_{a+1} \\ \vdots \\ \lambda_a\lambda_b \end{bmatrix}$$

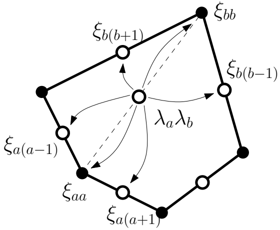
From quadratic to serendipity

We ***require*** the serendipity basis to have quadratic approximation power:

Constant precision: $$1 = \sum_i \xi_{ii} + 2\xi_{i(i+1)}$$

Linear precision: $$\mathbf{x} = \sum_i \mathbf{v}_i \xi_{ii} + 2\mathbf{v}_{i(i+1)} \xi_{i(i+1)}$$

Quadratic precision: $$\mathbf{x}\mathbf{x}^T = \sum_i \mathbf{v}_i \mathbf{v}_i^T \xi_{ii} + (\mathbf{v}_i \mathbf{v}_{i+1}^T + \mathbf{v}_{i+1} \mathbf{v}_i^T) \xi_{i(i+1)}$$



Theorem [Rand, G, Bajaj]

Constants $\{c_{ij}^{ab}\}$ exist for **any** convex polygon such that the resulting basis $\{\xi_{ij}\}$ satisfies constant, linear, and quadratic precision requirements.

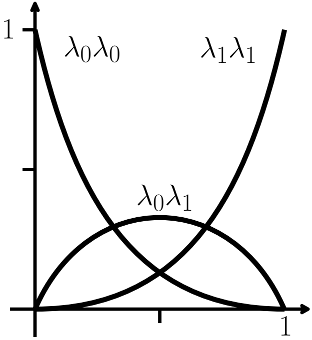
Proof: We produce a coefficient matrix $\mathbb{A}$ with the structure

$$\mathbb{A} := \left[\ \mathbb{I} \ \middle|\ \mathbb{A}'\ \right]$$

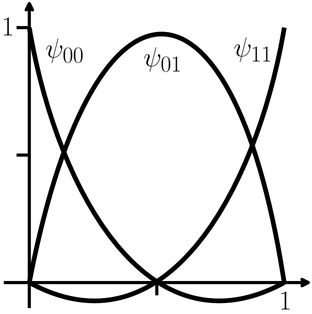
where $\mathbb{A}'$ has only six non-zero entries per column and show that the resulting functions satisfy the six precision equations.

Pairwise products vs. Lagrange basis

Even in 1D, pairwise products of barycentric functions do not form a Lagrange basis at interior degrees of freedom:



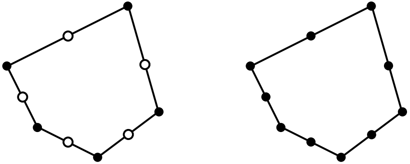
Pairwise products



Lagrange basis

Translation between these two bases is straightforward and generalizes to the higher dimensional case.

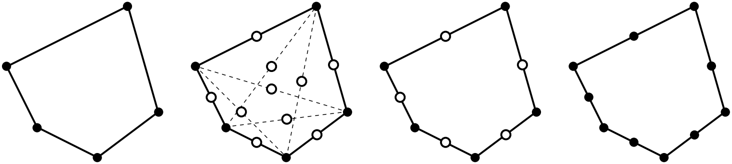
From serendipity to Lagrange



$$\{\xi_{ij}\} \xrightarrow{\mathbb{B}} \{\psi_{ij}\}$$

$$[\psi_{ij}] = \begin{bmatrix} \psi_{11} \\ \psi_{22} \\ \vdots \\ \psi_{nn} \\ \psi_{12} \\ \psi_{23} \\ \vdots \\ \psi_{n1} \end{bmatrix} = \left[\begin{array}{ccccc|ccccc} 1 & & & & & -1 & & \cdots & & -1 \\ & 1 & & & & -1 & -1 & \cdots & & \\ & & \ddots & & & & \ddots & \ddots & & \\ & & & \ddots & & & & \ddots & \ddots & \\ & & & & 1 & & & & -1 & -1 \\ \hline & & & & & 4 & & & & \\ & & & & & & 4 & & & \\ & & 0 & & & & & \ddots & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 4 \end{array}\right] \begin{bmatrix} \xi_{11} \\ \xi_{22} \\ \vdots \\ \xi_{nn} \\ \xi_{12} \\ \xi_{23} \\ \vdots \\ \xi_{n1} \end{bmatrix} = \mathbb{B}[\xi_{ij}].$$

Serendipity Theorem



$$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\lambda_a\lambda_b\} \xrightarrow{\mathbb{A}} \{\xi_{ij}\} \xrightarrow{\mathbb{B}} \{\psi_{ij}\}$$

Theorem [Rand, G, Bajaj]

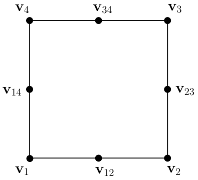
Given bounds on polygonal geometric quality:

- $||\mathbb{A}||$ is uniformly bounded,
- $||\mathbb{B}||$ is uniformly bounded, and
- span$\{\psi_{ij}\} \supset \mathcal{P}_2(\mathbb{R}^2)$ = quadratic polynomials in x and y

We obtain the quadratic *a priori* error estimate: $\boxed{||u - u_h||_{H^1(\Omega)} \leq C\, h^{\mathbf{2}}\, |u|_{H^3(\Omega)}}$

RAND, G, BAJAJ *Quadratic Serendipity Finite Element on Polygons Using Generalized Barycentric Coordinates*, Math. Comp., 2011

Special case of a square



Bilinear functions are barycentric coordinates:

$$\begin{aligned}\lambda_1 &= (1-x)(1-y)\\ \lambda_2 &= x(1-y)\\ \lambda_3 &= xy\\ \lambda_4 &= (1-x)y\end{aligned}$$

Compute $[\xi_{ij}] := \left[\ \mathbb{I}\ \middle|\ \mathbb{A}'\ \right][\lambda_a\lambda_b]$

$$\begin{bmatrix}\xi_{11}\\ \xi_{22}\\ \xi_{33}\\ \xi_{44}\\ \xi_{12}\\ \xi_{23}\\ \xi_{34}\\ \xi_{14}\end{bmatrix} = \begin{bmatrix}1 & & 0 & -1 & 0\\ 0 & & 0 & 0 & -1\\ 0 & & 0 & -1 & 0\\ 0 & \cdots & 0 & 0 & -1\\ 0 & \cdots & 0 & 1/2 & 1/2\\ 0 & & 0 & 1/2 & 1/2\\ 0 & & 0 & 1/2 & 1/2\\ 0 & & 1 & 1/2 & 1/2\end{bmatrix} \begin{bmatrix}\lambda_1\lambda_1\\ \lambda_2\lambda_2\\ \lambda_3\lambda_3\\ \lambda_4\lambda_4\\ \lambda_1\lambda_2\\ \lambda_2\lambda_3\\ \lambda_3\lambda_4\\ \lambda_1\lambda_4\end{bmatrix} = \begin{bmatrix}(1-x)(1-y)(1-x-y)\\ x(1-y)(x-y)\\ xy(-1+x+y)\\ (1-x)y(y-x)\\ (1-x)x(1-y)\\ x(1-y)y\\ (1-x)xy\\ (1-x)(1-y)y\end{bmatrix}$$

$$\operatorname{span}\left\{\xi_{ii}, \xi_{i(i+1)}\right\} = \operatorname{span}\left\{1, x, y, x^2, y^2, xy, x^2y, xy^2\right\} =: \mathcal{S}_2(I^2)$$

Hence, this provides a computational basis for the serendipity space $\mathcal{S}_2(I^2)$ defined in

Arnold, Awanou *The serendipity family of finite elements*, Found. Comp. Math, 2011.

Outline

1. Error estimates for linear case
2. Quadratic serendipity elements on polygons
3. **Basis construction for vector-valued problems**
4. Numerical results

From scalar to vector elements

The classical finite element sequences for a domain $\Omega \subset \mathbb{R}^n$ are written:

$$\begin{tikzcd}
n=2: & H^1 \arrow[r, "\mathrm{grad}"] & H(\mathrm{curl}) \arrow[r, leftrightarrow, "\mathrm{rot}"] & H(\mathrm{div}) \arrow[r, "\mathrm{div}"] & L^2 \\
n=3: & H^1 \arrow[r, "\mathrm{grad}"] & H(\mathrm{curl}) \arrow[r, "\mathrm{curl}"] & H(\mathrm{div}) \arrow[r, "\mathrm{div}"] & L^2
\end{tikzcd}$$

These correspond to the L^2 deRham diagrams from differential topology:

$$\begin{tikzcd}
n=2: & H\Lambda^0 \arrow[r, "d_0"] & H\Lambda^1 \arrow[r, leftrightarrow, "\cong"] & H\Lambda^1 \arrow[r, "d_1"] & H\Lambda^2 \\
n=3: & H\Lambda^0 \arrow[r, "d_0"] & H\Lambda^1 \arrow[r, "d_1"] & H\Lambda^2 \arrow[r, "d_2"] & H\Lambda^3
\end{tikzcd}$$

Conforming finite element subspaces of $H\Lambda^k$ are of two types:

$$\mathcal{P}_r\Lambda^k := k\text{-forms with degree } r \text{ polynomial coefficients}$$
$$\mathcal{P}_r^-\Lambda^k := \mathcal{P}_{r-1}\Lambda^k \oplus \{\text{certain additional } k\text{-forms}\}$$

This notation, from Finite Element Exterior Calculus, can be used to describe many well-known finite element spaces.

ARNOLD, FALK, WINTHER *Finite Element Exterior Calculus*, Bulletin of the AMS, 2010.

Classical finite element spaces on simplices

n=2 (triangles)

k	dim	space	type	classical description
0	3	$\mathcal{P}_1\Lambda^0$	H^1	Lagrange elements of degree ≤ 1
	3	$\mathcal{P}_1^-\Lambda^0$	H^1	Lagrange elements of degree ≤ 1
1	6	$\mathcal{P}_1\Lambda^1$	$H(\text{div})$	Brezzi-Douglas-Marini $H(\text{div})$ elements of degree ≤ 1
	3	$\mathcal{P}_1^-\Lambda^1$	$H(\text{div})$	Raviart-Thomas elements of order 0
2	3	$\mathcal{P}_1\Lambda^2$	L^2	discontinuous linear
	1	$\mathcal{P}_1^-\Lambda^2$	L^2	discontinuous piecewise constant

n=3 (tetrahedra)

0	4	$\mathcal{P}_1\Lambda^0$	H^1	Lagrange elements of degree ≤ 1
	4	$\mathcal{P}_1^-\Lambda^0$	H^1	Lagrange elements of degree ≤ 1
1	12	$\mathcal{P}_1\Lambda^1$	$H(\text{curl})$	Nédélec second kind $H(\text{curl})$ elements of degree ≤ 1
	6	$\mathcal{P}_1^-\Lambda^1$	$H(\text{curl})$	Nédélec first kind $H(\text{curl})$ elements of order 0
2	12	$\mathcal{P}_1\Lambda^2$	$H(\text{div})$	Nédélec second kind $H(\text{div})$ elements of degree ≤ 1
	4	$\mathcal{P}_1^-\Lambda^2$	$H(\text{div})$	Nédélec first kind $H(\text{div})$ elements of order 0
3	4	$\mathcal{P}_1\Lambda^3$	L^2	discontinuous linear
	1	$\mathcal{P}_1^-\Lambda^3$	L^2	discontinuous piecewise constant

Basis functions on simplices

n=2 **(triangles)**

k	dim	space	type	basis functions
0	3	$\mathcal{P}_1\Lambda^0$	H^1	λ_i
1	6	$\mathcal{P}_1\Lambda^1$	$H(\text{curl})$	$\lambda_i\nabla\lambda_j$
	6	$\mathcal{P}_1\Lambda^1$	$H(\text{div})$	$\text{rot}(\lambda_i\nabla\lambda_j)$
	3	$\mathcal{P}_1^-\Lambda^1$	$H(\text{curl})$	$\lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i$
	3	$\mathcal{P}_1^-\Lambda^1$	$H(\text{div})$	$\text{rot}(\lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i)$
2	3	$\mathcal{P}_1\Lambda^2$	L^2	piecewise linear functions
	1	$\mathcal{P}_1^-\Lambda^2$	L^2	piecewise constant functions

n=3 **(tetrahedra)**

0	4	$\mathcal{P}_1\Lambda^0$	H^1	λ_i
1	12	$\mathcal{P}_1\Lambda^1$	$H(\text{curl})$	$\lambda_i\nabla\lambda_j$
	6	$\mathcal{P}_1^-\Lambda^1$	$H(\text{curl})$	$\lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i$
2	12	$\mathcal{P}_1\Lambda^2$	$H(\text{div})$	$\lambda_i\nabla\lambda_j \times \nabla\lambda_k$
	4	$\mathcal{P}_1^-\Lambda^2$	$H(\text{div})$	$(\lambda_i\nabla\lambda_j \times \nabla\lambda_k) + (\lambda_j\nabla\lambda_k \times \nabla\lambda_i) + (\lambda_k\nabla\lambda_i \times \nabla\lambda_j)$
3	4	$\mathcal{P}_1\Lambda^3$	L^2	piecewise linear functions
	1	$\mathcal{P}_1^-\Lambda^3$	L^2	piecewise constant functions

Essential properties of basis functions

The vector-valued basis constructions ($0 < k < n$) have two key properties:

1 **Global continuity in $H(\mathrm{curl})$ or $H(\mathrm{div})$**

$\lambda_i \nabla \lambda_j$ agree on **tangential** components at element interfaces $\Longrightarrow$ $H(\mathrm{curl})$ continuity

$\lambda_i \nabla \lambda_j \times \nabla \lambda_k$ agree on **normal** components at element interfaces $\Longrightarrow$ $H(\mathrm{div})$ continuity

2 **Reproduction of requisite polynomial differential forms.**

For $i, j \in \{1, 2, 3\}$:

$$\mathrm{span}\{\lambda_i \nabla \lambda_j\} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x \end{bmatrix}, \begin{bmatrix} y \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \right\} \cong \mathcal{P}_1 \Lambda^1(\mathbb{R}^2)$$

$$\mathrm{span}\{\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i\} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right\} \cong \mathcal{P}_1^- \Lambda^1(\mathbb{R}^2)$$

Using generalized barycentric coordinates, we can extend all these results to polygonal and polyhedral elements.

Basis functions on polygons and polyhedra

Theorem [G., Rand, Bajaj]

Let P be a convex polygon or polyhedron. Given any set of generalized barycentric coordinates $\{\lambda_i\}$ associated to P, the functions listed below have **global continuity and** polynomial differential form **reproduction** properties as indicated.

n=2
(polygons)

k	space	type	functions
1	$\mathcal{P}_1\Lambda^1$	$H(\text{curl})$	$\lambda_i\nabla\lambda_j$
	$\mathcal{P}_1\Lambda^1$	$H(\text{div})$	$\text{rot}(\lambda_i\nabla\lambda_j)$
	$\mathcal{P}_1^-\Lambda^1$	$H(\text{curl})$	$\lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i$
	$\mathcal{P}_1^-\Lambda^1$	$H(\text{div})$	$\text{rot}(\lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i)$

n=3
(polyhedra)

1	$\mathcal{P}_1\Lambda^1$	$H(\text{curl})$	$\lambda_i\nabla\lambda_j$
	$\mathcal{P}_1^-\Lambda^1$	$H(\text{curl})$	$\lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i$
2	$\mathcal{P}_1\Lambda^2$	$H(\text{div})$	$\lambda_i\nabla\lambda_j \times \nabla\lambda_k$
	$\mathcal{P}_1^-\Lambda^2$	$H(\text{div})$	$(\lambda_i\nabla\lambda_j \times \nabla\lambda_k) + (\lambda_j\nabla\lambda_k \times \nabla\lambda_i) + (\lambda_k\nabla\lambda_i \times \nabla\lambda_j)$

Note: The indices range over ***all*** pairs or triples of vertex indices from P.

Polynomial differential form reproduction identities

Let $P \subset \mathbb{R}^3$ be a convex polyhedron with vertex set $\{\mathbf{v}_i\}$. Let $\mathbf{x} = \begin{bmatrix} x & y & z \end{bmatrix}^T$.

Then for any 3×3 real matrix $\mathbb{A}$,

$$\begin{aligned} \sum_{i,j} \lambda_i \nabla \lambda_j (\mathbf{v}_j - \mathbf{v}_i)^T &= \mathbb{I} \\ \sum_{i,j} (\mathbb{A}\mathbf{v}_i \cdot \mathbf{v}_j)(\lambda_i \nabla \lambda_j) &= \mathbb{A}\mathbf{x} \end{aligned}$$

$$\begin{aligned} \frac{1}{2} \sum_{i,j,k} \lambda_i \nabla \lambda_j \times \nabla \lambda_k \left((\mathbf{v}_j - \mathbf{v}_i) \times (\mathbf{v}_k - \mathbf{v}_i) \right)^T &= \mathbb{I} \\ \frac{1}{2} \sum_{i,j,k} (\mathbb{A}\mathbf{v}_i \cdot (\mathbf{v}_j \times \mathbf{v}_k))(\lambda_i \nabla \lambda_j \times \nabla \lambda_k) &= \mathbb{A}\mathbf{x}. \end{aligned}$$

By appropriate choice of constant entries for $\mathbb{A}$, the column vectors of $\mathbb{I}$ and $\mathbb{A}\mathbf{x}$ span $\mathcal{P}_1\Lambda^1 \subset H(\text{curl})$ or $\mathcal{P}_1\Lambda^2 \subset H(\text{div})$.

→ Additional identities for the remaining cases are stated in:

G, Rand, Bajaj *Construction of Scalar and Vector Finite Element Families on Polygonal and Polyhedral Meshes*, arXiv:1405.6978, 2014

Reducing the basis

In some cases, it should be possible to reduce the size of the basis constructed by our method, in an analogous fashion to the quadratic scalar case.

n=2 (polygons)

k	space	# construction	# boundary	# polynomial
0	$\mathcal{P}_1\Lambda^0(\mathfrak{m})/\mathcal{P}_0^-\Lambda^1(\mathfrak{m})$	v	v	3
1	$\mathcal{P}_1\Lambda^1(\mathfrak{m})$	$v(v-1)$	$2e$	6
	$\mathcal{P}_1^-\Lambda^1(\mathfrak{m})$	$\binom{v}{2}$	e	3
2	$\mathcal{P}_1\Lambda^2(\mathfrak{m})$	$\frac{v(v-1)(v-2)}{2}$	0	3
	$\mathcal{P}_1^-\Lambda^2(\mathfrak{m})$	$\binom{v}{3}$	0	1

→ The $n = 3$ (polyhedra) version of this table is given in:

G, RAND, BAJAJ *Construction of Scalar and Vector Finite Element Families on Polygonal and Polyhedral Meshes*, arXiv:1405.6978, 2014

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Matlab code for Wachspress coordinates on polygons

Input: The vertices $v_1, \ldots, v_n$ of a polygon and a point x

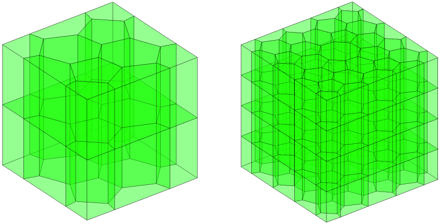
Output: Wachspress functions λ_i and their gradients $\nabla\lambda_i$

```
function [phi dphi] = wachspress2d(v,x)
n = size(v,1);
w = zeros(n,1);
R = zeros(n,2);
phi = zeros(n,1);
dphi = zeros(n,2);

un = getNormals(v); % computes the outward unit normal to each edge

p = zeros(n,2);
for i = 1:n
   h = dot(v(i,:) - x,un(i,:));
   p(i,:) = un(i,:) / h;
end

for i = 1:n
   im1 = mod(i-2,n) + 1;
   w(i) = det([p(im1,:);p(i,:)]);
   R(i,:) = p(im1,:) + p(i,:);
end

wsum = sum(w);
phi = w/wsum;

phiR = phi' * R;
for k = 1:2
   dphi(:,k) = phi .* (R(:,k) - phiR(:,k));
end
```

Matlab code for polygons *and polyhedra* (simple or non-simple) included in appendix of

FLOATER, G, SUKUMAR *Gradient bounds for Wachspress coordinates on polytopes*, SIAM J. Numerical Analysis, 2014.

Numerical results



h= 0.7071 h= 0.3955

→ We fix a sequence of polyhedral meshes where h denotes the maximum diameter of a mesh element.

→ $\exists\, \gamma > 0$ such that if any element from any mesh in the sequence is scaled to have diameter 1, the computed value of h_* will be $\geq \gamma$.

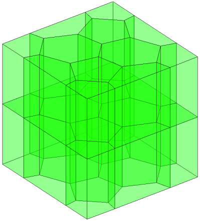
→ We solve the weak form of the Poisson problem:

$$\int_\Omega \nabla u \cdot \nabla w \, d\mathbf{x} = \int_\Omega f \; w \, d\mathbf{x}, \quad \forall w \in H_0^1(\Omega),$$

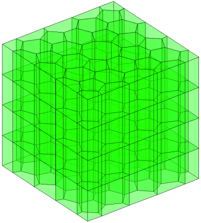
where $f(\mathbf{x})$ is defined so that the exact solution is $u(\mathbf{x}) = xyz(1-x)(1-y)(1-z)$.

→ Using Wachspress coordinates $\lambda_\mathbf{v}$, the local stiffness matrix has entries of the form $\displaystyle\int_P \nabla\lambda_\mathbf{v} \cdot \nabla\lambda_\mathbf{w} \, d\mathbf{x}$, which we integrate by tetrahedralizing P and using a second-order accurate quadrature rule (4 points per tetrahedron).

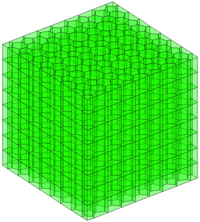
Numerical results



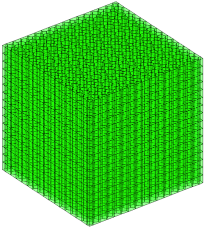
h= 0.7071



h= 0.3955



h= 0.1977

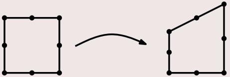


h= 0.0989

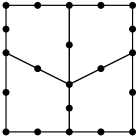
As expected, we observe optimal convergence convergence rates: quadratic in L^2 norm and linear in H^1 semi-norm.

Mesh	# of nodes	h	$\frac{\|\|u - u^h\|\|_{0,P}}{\|\|u\|\|_{0,P}}$	Rate	$\frac{\|u - u^h\|_{1,P}}{\|u\|_{1,P}}$	Rate
a	78	0.7071	2.0×10^{-1}	–	4.1×10^{-1}	–
b	380	0.3955	5.4×10^{-2}	2.28	2.1×10^{-1}	1.14
c	2340	0.1977	1.4×10^{-2}	1.96	1.1×10^{-1}	0.97
d	16388	0.0989	3.5×10^{-3}	1.99	5.4×10^{-2}	0.99
e	122628	0.0494	8.8×10^{-4}	**2.00**	2.7×10^{-2}	**0.99**

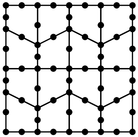
Numerical evidence for non-affine image of a square

Instead of mapping  ,
use quadratic serendipity **GBC** interpolation with mean value coordinates:

$$u_h = I_q u := \sum_{i=1}^{n} u(\mathbf{v}_i)\psi_{ii} + u\left(\frac{\mathbf{v}_i + \mathbf{v}_{i+1}}{2}\right)\psi_{i(i+1)}$$



$n = 2$



$n = 4$

Non-affine bilinear mapping

	$\|\|u - u_h\|\|_{L^2}$		$\|\|\nabla(u - u_h)\|\|_{L^2}$	
n	error	rate	error	rate
2	5.0e-2		6.2e-1	
4	6.7e-3	2.9	1.8e-1	1.8
8	9.7e-4	2.8	5.9e-2	1.6
16	1.6e-4	2.6	2.3e-2	1.4
32	3.3e-5	2.3	1.0e-2	1.2
64	7.4e-6	2.1	4.96e-3	1.1

ARNOLD, BOFFI, FALK, Math. Comp., 2002

Quadratic serendipity **GBC** method

	$\|\|u - u_h\|\|_{L^2}$		$\|\|\nabla(u - u_h)\|\|_{L^2}$	
n	error	rate	error	rate
2	2.34e-3		2.22e-2	
4	3.03e-4	2.95	6.10e-3	1.87
8	3.87e-5	2.97	1.59e-3	1.94
16	4.88e-6	2.99	4.04e-4	1.97
32	6.13e-7	3.00	1.02e-4	1.99
64	7.67e-8	3.00	2.56e-5	1.99
128	9.59e-9	3.00	6.40e-6	2.00
256	1.20e-9	3.00	1.64e-6	1.96

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Michael Floater	University of Oslo
N. Sukumar	UC Davis

Thanks for the invitation to speak!

Slides and pre-prints: `http://math.arizona.edu/~agillette/`

More on GBCs: `http://www.inf.usi.ch/hormann/barycentric`