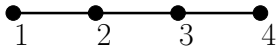
An Introduction to Hermite Serendipity Finite Element Methods

Andrew Gillette

Department of Mathematics
University of California, San Diego

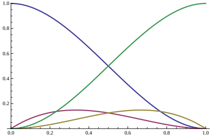
http://ccom.ucsd.edu/∼agillette/

Cubic Hermite Geometric Decomposition: 1D

monomials	approxim.	basis functions	geometry	domain points
$\{1, x, x^2, x^3\}$	$\longleftrightarrow$	$\{\psi_1, \psi_2, \psi_3, \psi_4\}$	$\longleftrightarrow$	

Cubic Hermite Basis on $[0, 1]$

$$\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{bmatrix} := \begin{bmatrix} 1 - 3x^2 + 2x^3 \\ x - 2x^2 + x^3 \\ x^2 - x^3 \\ 3x^2 - 2x^3 \end{bmatrix}$$



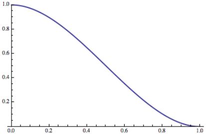
Approximation: $x^r = \sum_{i=1}^{4} \varepsilon_{r,i}\psi_i$, for $r = 0, 1, 2, 3$, where $[\varepsilon_{r,i}] = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -3 & 1 \end{pmatrix}$

Geometry: If $a(x)$ is a cubic polynomial then:

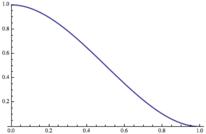
$$a(x) = \underbrace{a(0)}_{\text{value}}\psi_1 + \underbrace{a'(0)}_{\text{derivative}}\psi_2 - \underbrace{a'(1)}_{\text{derivative}}\psi_3 + \underbrace{a(1)}_{\text{value}}\psi_4$$

Tensor Product Polynomials

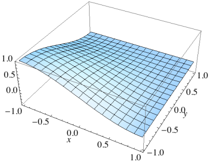
We can use our 1D Hermite functions to make 2D Hermite functions:



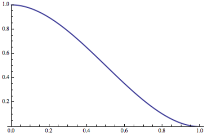
$\times$



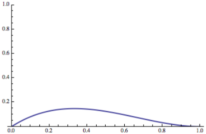
$=$



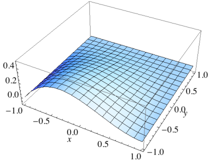
$\psi_1(x) \times \psi_1(y) = \psi_{11}(x, y)$



$\times$



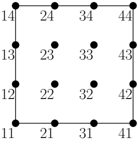
$=$



$\psi_1(x) \times \psi_2(y) = \psi_{12}(x, y)$

Cubic Hermite Geometric Decomposition: 2D

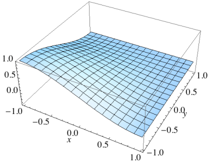
$$\left\{ \begin{array}{c} x^r y^s \\ 0 \leq r, s \leq 3 \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \psi_i(x)\psi_j(y) \\ 1 \leq i, j \leq 4 \end{array} \right\} \quad \longleftrightarrow$$



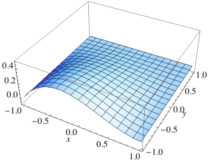
monomials $\longleftrightarrow$ **basis functions** $\longleftrightarrow$ **domain points**

Approximation: $x^r y^s = \displaystyle\sum_{i=1}^{4} \sum_{j=1}^{4} \varepsilon_{r,i}\varepsilon_{s,j}\psi_{ij}$, for $0 \leq r, s \leq 3$, $\varepsilon_{r,i}$ as in 1D.

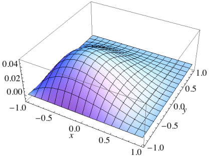
Geometry:



ψ_{11}



ψ_{21}

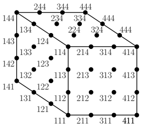


ψ_{22}

$$a(x,y) = \underbrace{a|_{(0,0)}}_{\text{value}} \psi_{11} + \underbrace{\partial_x a|_{(0,0)}}_{\text{1st deriv.}} \psi_{21} + \underbrace{\partial_y a|_{(0,0)}}_{\text{1st deriv.}} \psi_{12} + \underbrace{\partial_x \partial_y a|_{(0,0)}}_{\text{2nd deriv.}} \psi_{22} + \cdots$$

Cubic Hermite Geometric Decomposition: 3D

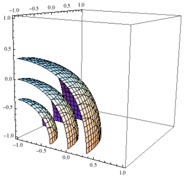
$$\left\{ \begin{array}{c} x^r y^s z^t \\ 0 \le r, s, t \le 3 \end{array} \right\} \longleftrightarrow \begin{array}{c} \psi_i(x)\psi_j(y)\psi_k(z) \\ 1 \le i, j, k \le 4 \end{array} \longleftrightarrow$$



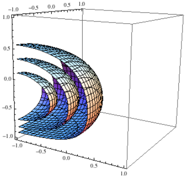
monomials $\longleftrightarrow$ **basis functions** $\longleftrightarrow$ **domain points**

Approximation: $x^r y^s z^t = \displaystyle\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4} \varepsilon_{r,i}\varepsilon_{s,j}\varepsilon_{t,k}\psi_{ijk}$, for $0 \le r, s, t \le 3$, $\varepsilon_{r,i}$ as in 1D.

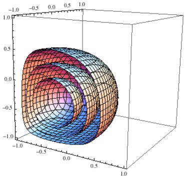
Geometry: Contours of level sets of the basis functions:



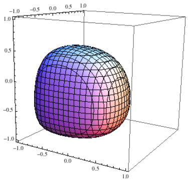
ψ_{111}



ψ_{112}



ψ_{212}



ψ_{222}

Which monomials do we really need?

$$\text{degree}(x^r y^s) = r + s$$

$$\text{superlinear degree}(x^r y^s) = r + s - \{\text{\# of linearly appearing variables}\}$$

	degree	superlinear degree
xy^2		
x^3y		
xy^3		
x^2y^2		
x^3y^2		

Which monomials do we really need?

$$\text{degree}(x^r y^s) = r + s$$

$$\text{superlinear degree}(x^r y^s) = r + s - \{\text{\# of linearly appearing variables}\}$$

	degree	superlinear degree
xy^2	3	2
x^3y		
xy^3		
x^2y^2		
x^3y^2		

Which monomials do we really need?

$$\text{degree}(x^r y^s) = r + s$$

$$\text{superlinear degree}(x^r y^s) = r + s - \{\text{\# of linearly appearing variables}\}$$

	degree	superlinear degree
xy^2	3	2
x^3y	4	3
xy^3		
x^2y^2		
x^3y^2		

Which monomials do we really need?

$$\text{degree}(x^r y^s) = r + s$$

$$\text{superlinear degree}(x^r y^s) = r + s - \{\#\text{ of linearly appearing variables}\}$$

	degree	superlinear degree
xy^2	3	2
x^3y	4	3
xy^3	4	3
x^2y^2		
x^3y^2		

Which monomials do we really need?

$$\text{degree}(x^r y^s) = r + s$$

$$\text{superlinear degree}(x^r y^s) = r + s - \{\text{\# of linearly appearing variables}\}$$

	degree	superlinear degree
xy^2	3	2
x^3y	4	3
xy^3	4	3
x^2y^2	4	4
x^3y^2		

Which monomials do we really need?

$$\text{degree}(x^r y^s) = r + s$$

$$\text{superlinear degree}(x^r y^s) = r + s - \{\text{\# of linearly appearing variables}\}$$

	degree	superlinear degree
xy^2	3	2
x^3y	4	3
xy^3	4	3
x^2y^2	4	4
x^3y^2	5	5

regular cubics (dim=16)

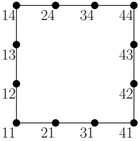
$$\{\underbrace{1, x, y, x^2, y^2, xy, x^3, y^3, x^2y, xy^2, x^3y, xy^3}_{\text{superlinear cubics (dim=12)}}, x^2y^2, x^3y^2, x^2y^3, x^3y^3\}$$

superlinear cubics (dim=12)

Recent mathematical research shows that degree r polynomials and **superlinear** degree r polynomials have the **same** approximation power for the models we are studying.

Theorem [G, 2012]: A Hermite-like geometric decomposition of $\mathcal{S}_3([0,1]^2)$ exists.

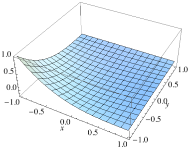
$$\underbrace{\left\{ \begin{array}{c} x^r y^s \\ \text{sldeg} \leq 3 \end{array} \right\}}_{\mathcal{S}_3([0,1]^2)} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \vartheta_{\ell m} \\ \text{(to be defined)} \end{array} \right\} \quad \longleftrightarrow$$



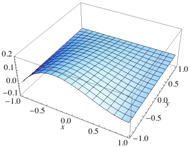
monomials $\longleftrightarrow$ **basis functions** $\longleftrightarrow$ **domain points**

Approximation: $x^r y^s = \sum_{\ell m} \varepsilon_{r,i} \varepsilon_{s,j} \vartheta_{\ell m}$, for superlinear degree$(x^r y^s) \leq 3$

Geometry:



ϑ_{11}

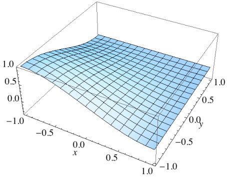


ϑ_{21}

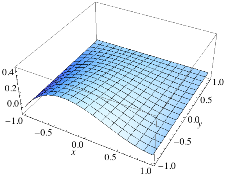
$$\begin{aligned} a(x,y) &= a|_{(0,0)} \vartheta_{11} \\ &+ \partial_x a|_{(0,0)} \vartheta_{21} \\ &+ \partial_y a|_{(0,0)} \vartheta_{12} \\ &+ \cdots \end{aligned}$$

Hermite Style Serendipity Functions (2D)

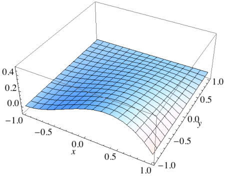
$$[\vartheta_{\ell m}] = \begin{bmatrix} \vartheta_{11} \\ \vartheta_{14} \\ \vartheta_{41} \\ \vartheta_{44} \\ \vartheta_{12} \\ \vartheta_{13} \\ \vartheta_{42} \\ \vartheta_{43} \\ \vartheta_{21} \\ \vartheta_{31} \\ \vartheta_{24} \\ \vartheta_{34} \end{bmatrix} = \begin{bmatrix} -(x-1)(y-1)(-2+x+x^2+y+y^2) \\ (x-1)(y+1)(-2+x+x^2-y+y^2) \\ (x+1)(y-1)(-2-x+x^2+y+y^2) \\ -(x+1)(y+1)(-2-x+x^2-y+y^2) \\ -(x-1)(y-1)^2(y+1) \\ (x-1)(y-1)(y+1)^2 \\ (x+1)(y-1)^2(y+1) \\ -(x+1)(y-1)(y+1)^2 \\ -(x-1)^2(x+1)(y-1) \\ (x-1)(x+1)^2(y-1) \\ (x-1)^2(x+1)(y+1) \\ -(x-1)(x+1)^2(y+1) \end{bmatrix} \cdot \frac{1}{8}$$



ϑ_{11}



ϑ_{21}

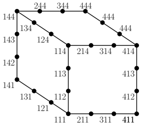


ϑ_{31}

Cubic Hermite Serendipity Geom. Decomp: 3D

Theorem [G, 2012]: A Hermite-like geometric decomposition of $\mathcal{S}_3([0,1]^3)$ exists.

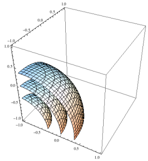
$$\underbrace{\left\{ \begin{array}{c} x^r y^s z^t \\ \text{sldeg} \leq 3 \end{array} \right\}}_{\mathcal{S}_3([0,1]^3)} \longleftrightarrow \left\{ \begin{array}{c} \vartheta_{\ell mn} \\ \text{(to be defined)} \end{array} \right\} \longleftrightarrow$$



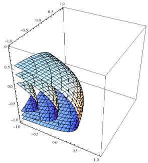
monomials $\longleftrightarrow$ **basis functions** $\longleftrightarrow$ **domain points**

Approximation: $x^r y^s z^t = \sum_{\ell mn} \varepsilon_{r,i}\varepsilon_{s,j}\varepsilon_{t,k}\vartheta_{\ell mn}$, for superlinear degree$(x^r y^s z^t) \leq 3$

Geometry:



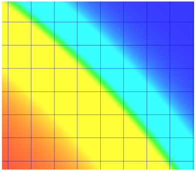
ϑ_{111}

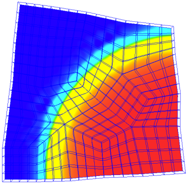


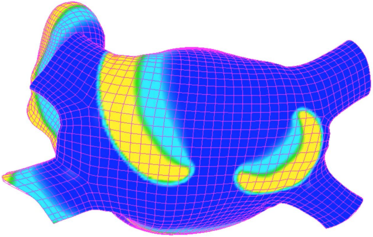
ϑ_{112}

$$\begin{aligned} a(x,y,z) &= a|_{(0,0,0)}\vartheta_{111} \\ &+ \partial_x a|_{(0,0,0)}\vartheta_{211} \\ &+ \partial_y a|_{(0,0,0)}\vartheta_{121} \\ &+ \partial_z a|_{(0,0,0)}\vartheta_{112} \\ &+ \cdots \end{aligned}$$

Application: Cardiac Electrophysiology







→ Cubic Hermite serendipity functions recently incorporated into Continuity software package for cardiac electrophysiology models.

→ Used to solve the *monodomain* equations, a type of reaction-diffusion equation

→ Initial results show agreement of serendipity and standard tricubics on benchmark problem with a **4x computational speedup** in 3D.

→ Fast computation essential to clinical applications and 'real time' simulations

GONZALEZ, VINCENT, G., MCCULLOCH *High Order Interpolation Methods in Cardiac Electrophysiology Simulation*, in preparation, 2013.

Acknowledgments

Michael Holst	UC San Diego
National Biomedical Computation Resource	UC San Diego
Matt Gonzalez	UC San Diego
Kevin Vincent	UC San Diego
Andrew McCulloch	UC San Diego

Thanks for the opportunity to speak!

Slides and pre-prints: `http://ccom.ucsd.edu/~agillette`