Polygonal, Polyhedral, and Serendipity Finite Element Methods

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What are *a priori* FEM error estimates?

Poisson's equation in $\mathbb{R}^n$: Given a domain $\mathcal{D} \subset \mathbb{R}^n$ and $f : \mathcal{D} \to \mathbb{R}$, find u such that

$$\begin{array}{rrcll} \text{strong form} & -\Delta u & = & f & u \in H^2(\mathcal{D}) \\ \text{weak form} & \displaystyle\int_{\mathcal{D}} \nabla u \cdot \nabla \phi & = & \displaystyle\int_{\mathcal{D}} f\,\phi & \forall \phi \in H^1(\mathcal{D}) \\ \text{discrete form} & \displaystyle\int_{\mathcal{D}} \nabla u_h \cdot \nabla \phi_h & = & \displaystyle\int_{\mathcal{D}} f\,\phi_h & \forall \phi_h \in V_h \leftarrow \textit{finite dim.} \subset H^1(\mathcal{D}) \end{array}$$

Typical **finite element method:**

$\to$ Mesh $\mathcal{D}$ by polytopes $\{\Omega\}$ with vertices $\{\mathbf{v}_i\}$; define $h := \max \operatorname{diam}(\Omega)$.

$\to$ Fix $\boxed{\text{basis functions } \lambda_i}$ with local piecewise support, e.g. barycentric functions.

$\to$ Define u_h such that it uses the λ_i to approximate u, e.g. $\boxed{u_h := \sum_i u(\mathbf{v}_i)\lambda_i}$

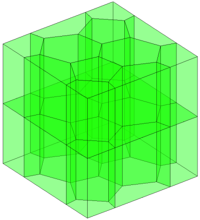
A linear system for u_h can then be derived, admitting an ***a priori* error estimate:**

$$\underbrace{||u - u_h||_{H^1(\Omega)}}_{\text{approximation error}} \leq \underbrace{C\, h^{\mathbf{p}}\, |u|_{H^{\mathbf{p+1}}(\Omega)}}_{\text{optimal error bound}}, \quad \forall u \in H^{\mathbf{p+1}}(\Omega),$$

provided that the λ_i span all degree p polynomials on each polytope Ω.

Two trends in contemporary finite element research

1 **Polygonal / polyhedral domain meshes**

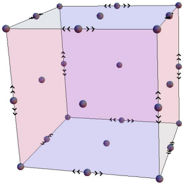


$\rightarrow$ Greater geometric flexibility can alleviate known difficulties with simplicial and cubical elements.

TALISCHI, PAULINO, PEREIRA, MENEZES, *PolyMesher: a general-purpose mesh generator for polygonal elements written in Matlab.* Structural and Multidisciplinary Optimization, 2012.

SUKUMAR *Quadratic maximum-entropy serendipity shape functions for arbitrary planar polygons.* Computer Methods in Applied Mechanics and Engineering, 2013.

2 **'Serendipity' higher order methods**



$\rightarrow$ Long observed but only recently formalized theory ensuring order p function approximation with many fewer basis functions than 'expected.'

ARNOLD, AWANOU *The serendipity family of finite elements*, Found. Comp. Math, 2011.

DA VEIGA, BREZZI, CANGIANI, MANZINI, RUSSO *Basic Principles of Virtual Element Methods*, M3AS, 2013.

$$||u - u_h||_{H^1(\Omega)} \leq C\, h^{\mathbf{p}}\, |u|_{H^{\mathbf{p+1}}(\Omega)}, \quad \forall u \in H^{\mathbf{p+1}}(\Omega)$$

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4. Future directions

The generalized barycentric coordinate approach

Let $\Omega \subset \mathbb{R}^2$ be a convex polygon with vertex set V. We say that a set of functions

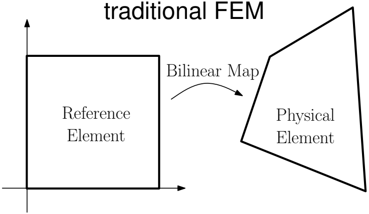
$$\lambda_{\mathbf{v}} : \Omega \to \mathbb{R} \quad \text{are} \quad \textbf{generalized barycentric coordinates (GBCs) on } \Omega$$

if they satisfy $\quad \lambda_{\mathbf{v}} \geq 0$ on Ω and $L = \sum_{\mathbf{v} \in V} L(\mathbf{v_v})\lambda_{\mathbf{v}}, \quad \forall\, L : \Omega \to \mathbb{R}$ linear.

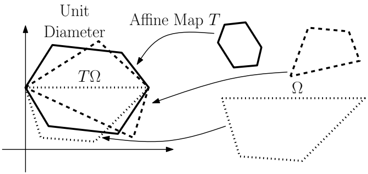
Familiar properties are implied by this definition:

$$\underbrace{\sum_{\mathbf{v} \in V} \lambda_{\mathbf{v}} \equiv 1}_{\text{partition of unity}} \qquad \underbrace{\sum_{\mathbf{v} \in V} \mathbf{v}\lambda_{\mathbf{v}}(\mathbf{x}) = \mathbf{x}}_{\text{linear precision}} \qquad \underbrace{\lambda_{\mathbf{v}_i}(\mathbf{v}_j) = \delta_{ij}}_{\text{interpolation}}$$

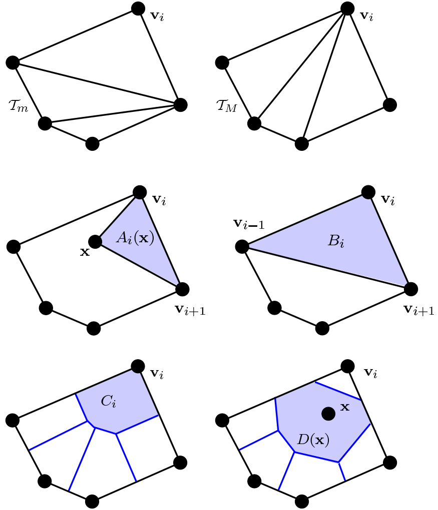
traditional FEM



family of **GBC** reference elements



Many generalizations to choose from . . .



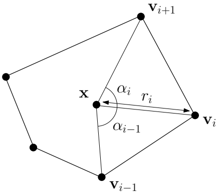
- Triangulation
  $\Rightarrow$ Floater, Hormann, Kós, *A general construction of barycentric coordinates over convex polygons*, 2006

  $$0 \leq \lambda_i^{\mathcal{T}_m}(\mathbf{x}) \leq \lambda_i(\mathbf{x}) \leq \lambda_i^{\mathcal{T}_M}(\mathbf{x}) \leq 1$$

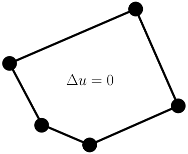
- Wachspress
  $\Rightarrow$ Wachspress, *A Rational Finite Element Basis*, 1975.
  $\Rightarrow$ Warren, *Barycentric coordinates for convex polytopes*, 1996.

- Sibson / Laplace
  $\Rightarrow$ Sibson, *A vector identity for the Dirichlet tessellation*, 1980.
  $\Rightarrow$ Hiyoshi, Sugihara, *Voronoi-based interpolation with higher continuity*, 2000.

Many generalizations to choose from . . .



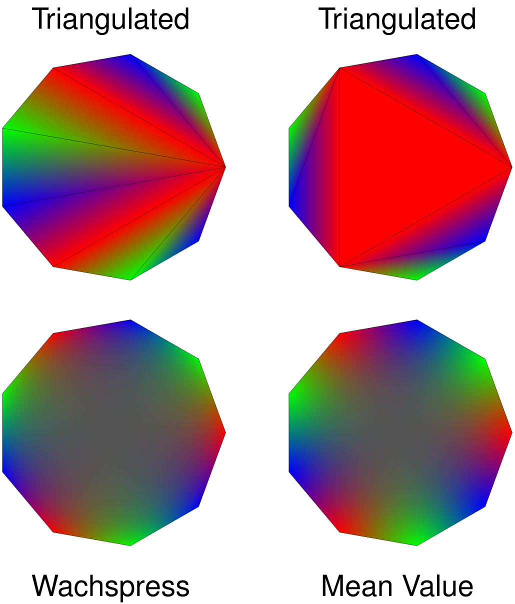
- Mean value
  $\Rightarrow$ FLOATER, *Mean value coordinates*, 2003.
  $\Rightarrow$ FLOATER, KÓS, REIMERS, *Mean value coordinates in 3D*, 2005.



- Harmonic
  $\Rightarrow$ WARREN, SCHAEFER, HIRANI, DESBRUN, *Barycentric coordinates for convex sets*, 2007.
  $\Rightarrow$ CHRISTIANSEN, *A construction of spaces of compatible differential forms on cellular complexes*, 2008.

Many more papers could be cited (maximum entropy coordinates, moving least squares coordinates, surface barycentric coordinates, etc...)

Comparison via ‘eyeball’ norm



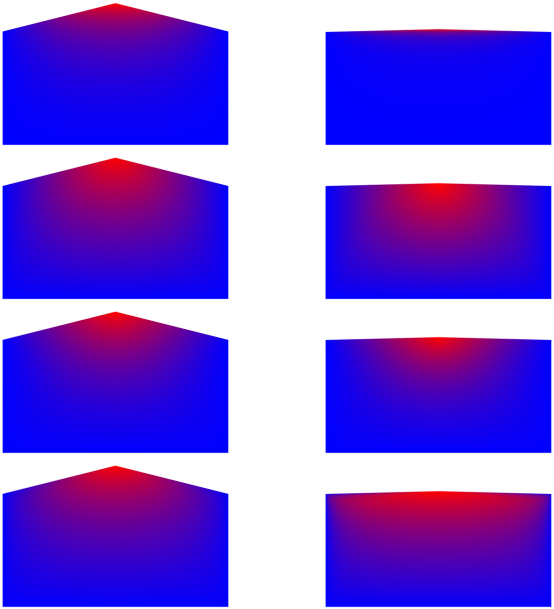
Comparison via ‘eyeball’ norm

Wachspress

Sibson

Mean Value

Discrete Harmonic



Optimal convergence estimates on polygons

Let Ω be a convex polygon with vertex set V.

For linear elements, an **optimal convergence estimate** has the form

$$\underbrace{\left|\left| u - \sum_{\mathbf{v}\in V} u(\mathbf{v})\lambda_{\mathbf{v}} \right|\right|_{H^1(\Omega)}}_{\text{approximation error}} \leq \underbrace{C \, \mathrm{diam}(\Omega) \, |u|_{H^2(\Omega)}}_{\text{optimal error bound}}, \quad \forall u \in H^2(\Omega). \tag{1}$$

The **Bramble-Hilbert lemma** in this context says that any $u \in H^2(\Omega)$ is close to a first order polynomial in H^1 norm.

VERFÜRTH, *A note on polynomial approximation in Sobolev spaces*, Math. Mod. Num. An., 2008.
DEKEL, LEVIATAN, *The Bramble-Hilbert lemma for convex domains*, SIAM J. Math. An., 2004.

For (1), it suffices to prove an **H^1-interpolant estimate** over domains of diameter one:

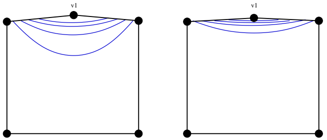
$$\left|\left| \sum_{\mathbf{v}\in V} u(\mathbf{v})\lambda_{\mathbf{v}} \right|\right|_{H^1(\Omega)} \leq C_I \, |u|_{H^2(\Omega)}, \quad \forall u \in H^1(\Omega). \tag{2}$$

For (2), it suffices to **bound the gradients** of the $\{\lambda_{\mathbf{v}}\}$, i.e. prove $\exists \, C_\lambda \in \mathbb{R}$ such that

$$||\nabla \lambda_{\mathbf{v}}||_{L^2(\Omega)} \leq C_\lambda. \tag{3}$$

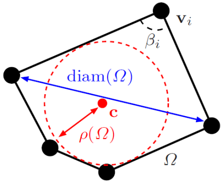
Geometric criteria for convergence estimates

To bound the gradients of the coordinates, we need control of the element geometry.



Let $\rho(\Omega)$ denote the radius of the largest inscribed circle.
The **aspect ratio** γ is defined by

$$\gamma = \frac{\mathrm{diam}(\Omega)}{\rho(\Omega)} \in (2, \infty)$$



Three possible geometric conditions on a polygonal mesh:

G1.	BOUNDED ASPECT RATIO:	$\exists\ \gamma^* < \infty$	such that	$\gamma < \gamma^*$
G2.	MINIMUM EDGE LENGTH:	$\exists\ d_* > 0$	such that	$\lvert\mathbf{v}_i - \mathbf{v}_{i-1}\rvert > d_*$
G3.	MAXIMUM INTERIOR ANGLE:	$\exists\ \beta^* < \pi$	such that	$\beta_i < \beta^*$

Polygonal Finite Element Optimal Convergence

Theorem

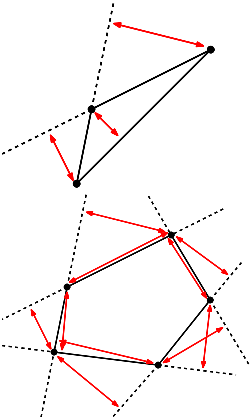
In the table, any necessary geometric criteria to achieve the ***a priori* linear error estimate** are denoted by N. The set of geometric criteria denoted by S in each row ***taken together*** are sufficient to guarantee the estimate.

		G1 (aspect ratio)	**G2** (min edge length)	**G3** (max interior angle)
Triangulated	λ^{Tri}	-	-	S,N
Wachspress	λ^{Wach}	S	S	S,N
Sibson	λ^{Sibs}	S	S	-
Mean Value	λ^{MV}	S	S	-
Harmonic	λ^{Har}	S	-	-

G, Rand, Bajaj *Error Estimates for Generalized Barycentric Interpolation* Advances in Computational Mathematics, 37:3, 417-439, 2012

Rand, G, Bajaj *Interpolation Error Estimates for Mean Value Coordinates*, Advances in Computational Mathematics, 39:2, 327-347, 2013.

A geometric criterion for polyhedra



Observe that on triangles of fixed diameter:

$$|\nabla\lambda| \text{ large} \iff \exists \text{ large interior angle}$$
$$\iff \exists \text{ small altitude 'at a vertex'}$$

For Wachspress coordinates, we generalize to polygons:

$$|\nabla\lambda^{\text{Wach}}| \text{ large} \iff \exists \text{ small 'altitude' at a vertex}$$

and then to polyhedra:

Given a simple, convex d-dimensional polytope $P \subset \mathbb{R}^d$, let

$h_* :=$ minimum distance from a vertex to a hyper-plane of a non-incident face.

Then $\quad |\nabla\lambda^{\text{Wach}}| \text{ large} \iff h_* \text{ small}$

Upper and lower bounds on polytopes

On a polytope $P \subset \mathbb{R}^n$, define $\Lambda := \sup_{\mathbf{x}\in P} \sum_{\mathbf{v}} |\nabla \lambda_{\mathbf{v}}^{\text{Wach}}(\mathbf{x})|$.

simple convex polytope in $\mathbb{R}^n$	$\frac{1}{h_*}$	$\leq$	Λ	$\leq$	$\frac{2n}{h_*}$
n-simplex in $\mathbb{R}^n$	$\frac{1}{h_*}$	$\leq$	Λ	$\leq$	$\frac{n+1}{h_*}$
hyper-rectangle in $\mathbb{R}^n$	$\frac{1}{h_*}$	$\leq$	Λ	$\leq$	$\frac{n+\sqrt{n}}{h_*}$
regular k-gon in $\mathbb{R}^2$	$\frac{2(1+\cos(\pi/k))}{h_*}$	$\leq$	Λ	$\leq$	$\frac{4}{h_*}$

Note that $\lim_{k\to\infty} 2(1+\cos(\pi/k)) = 4$, so the bound is **sharp** in $\mathbb{R}^2$.

FLOATER, G, SUKUMAR *Gradient bounds for Wachspress coordinates on polytopes*, SIAM J. Numerical Analysis, 2014.

Outline

1. Linear elements on polygons
2. Quadratic serendipity elements on polygons
3. Order r serendipity elements on n-cubes
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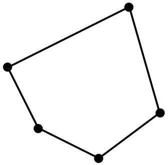
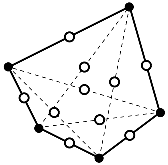
From linear to quadratic elements

A naïve quadratic element is formed by products of linear **GBCs**:



$$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\lambda_a\lambda_b\}$$



Why is this naïve?

- For a k-gon, this construction gives $k + \binom{k}{2}$ basis functions $\lambda_a\lambda_b$
- The space of quadratic polynomials is only dimension 6: $\{1, x, y, xy, x^2, y^2\}$
- Conforming to a linear function on the boundary requires 2 degrees of freedom per edge $\Rightarrow$ *only $2k$ functions needed!*

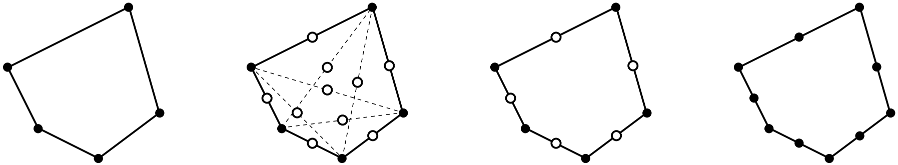
Problem Statement

Construct $2k$ basis functions associated to the vertices and edge midpoints of an arbitrary k-gon such that a quadratic convergence estimate is obtained.

Polygonal Quadratic Serendipity Elements

We define matrices $\mathbb{A}$ and $\mathbb{B}$ to reduce the naïve quadratic basis.

$$\begin{aligned}
\textbf{filled dot} &= \textbf{Lagrangian}\text{ domain point}\\
&= \text{all functions in the set evaluate to 0}\\
&\qquad \text{except the associated function which evaluates to 1}\\
\textbf{open dot} &= \text{non-Lagrangian domain point}\\
&= \text{partition of unity satisfied, but not Lagrange property}
\end{aligned}$$



$$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\lambda_a\lambda_b\} \xrightarrow{\mathbb{A}} \{\xi_{ij}\} \xrightarrow{\mathbb{B}} \{\psi_{ij}\}$$

k	$k+\binom{k}{2}$	$2k$	$2k$
Linear	Quadratic	Serendipity	Lagrange

From quadratic to serendipity

The bases are ordered as follows:

ξ_{ii}	and	$\lambda_a\lambda_a$	= basis functions associated with vertices
$\xi_{i(i+1)}$	and	$\lambda_a\lambda_{a+1}$	= basis functions associated with edge midpoints
		$\lambda_a\lambda_b$	= basis functions associated with interior diagonals, i.e. $b \notin \{a-1, a, a+1\}$

Serendipity basis functions ξ_{ij} are a linear combination of pairwise products $\lambda_a\lambda_b$:

$$\begin{bmatrix} \xi_{ii} \\ \vdots \\ \xi_{i(i+1)} \end{bmatrix} = \mathbb{A} \begin{bmatrix} \lambda_a\lambda_a \\ \vdots \\ \lambda_a\lambda_{a+1} \\ \vdots \\ \lambda_a\lambda_b \end{bmatrix} = \begin{bmatrix} c_{11}^{11} & \cdots & c_{ab}^{11} & \cdots & c_{(n-2)n}^{11} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{11}^{ij} & \cdots & c_{ab}^{ij} & \cdots & c_{(n-2)n}^{ij} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{11}^{n(n+1)} & \cdots & c_{ab}^{n(n+1)} & \cdots & c_{(n-2)n}^{n(n+1)} \end{bmatrix} \begin{bmatrix} \lambda_a\lambda_a \\ \vdots \\ \lambda_a\lambda_{a+1} \\ \vdots \\ \lambda_a\lambda_b \end{bmatrix}$$

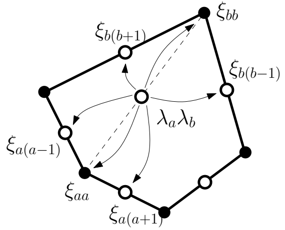
From quadratic to serendipity

We ***require*** the serendipity basis to have quadratic approximation power:

Constant precision: $\quad 1 = \sum_i \xi_{ii} + 2\xi_{i(i+1)}$

Linear precision: $\quad \mathbf{x} = \sum_i \mathbf{v}_i \xi_{ii} + 2\mathbf{v}_{i(i+1)} \xi_{i(i+1)}$

Quadratic precision: $\quad \mathbf{x}\mathbf{x}^T = \sum_i \mathbf{v}_i \mathbf{v}_i^T \xi_{ii} + (\mathbf{v}_i \mathbf{v}_{i+1}^T + \mathbf{v}_{i+1} \mathbf{v}_i^T) \xi_{i(i+1)}$



Theorem

Constants $\{c_{ij}^{ab}\}$ exist for **any** convex polygon such that the resulting basis $\{\xi_{ij}\}$ satisfies constant, linear, and quadratic precision requirements.

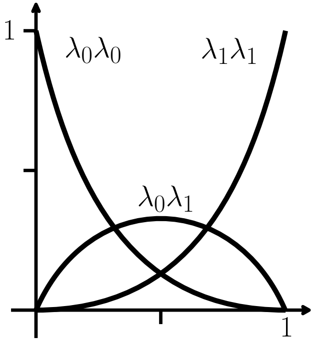
Proof: We produce a coefficient matrix $\mathbb{A}$ with the structure

$$\mathbb{A} := \left[\ \mathbb{I} \ \middle| \ \mathbb{A}' \ \right]$$

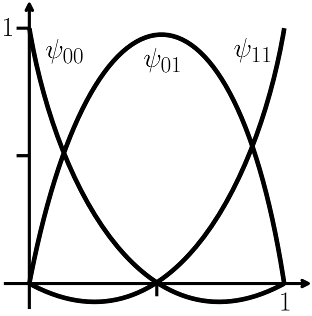
where $\mathbb{A}'$ has only six non-zero entries per column and show that the resulting functions satisfy the six precision equations.

Pairwise products vs. Lagrange basis

Even in 1D, pairwise products of barycentric functions do not form a Lagrange basis at interior degrees of freedom:



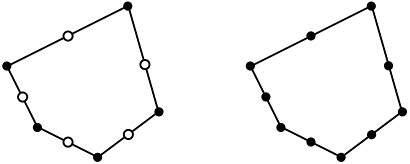
Pairwise products



Lagrange basis

Translation between these two bases is straightforward and generalizes to the higher dimensional case.

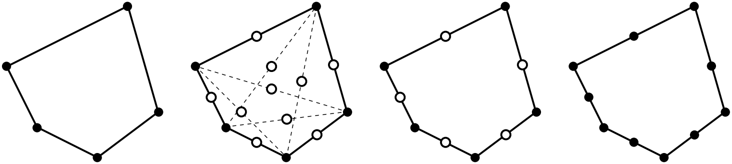
From serendipity to Lagrange



$$\{\xi_{ij}\} \xrightarrow{\quad\mathbb{B}\quad} \{\psi_{ij}\}$$

$$[\psi_{ij}] = \begin{bmatrix} \psi_{11} \\ \psi_{22} \\ \vdots \\ \psi_{nn} \\ \psi_{12} \\ \psi_{23} \\ \vdots \\ \psi_{n1} \end{bmatrix} = \left[\begin{array}{ccccc|ccccc} 1 & & & & & -1 & & \cdots & & -1 \\ & 1 & & & & -1 & -1 & \cdots & & \\ & & \ddots & & & & \ddots & \ddots & & \\ & & & \ddots & & & & \ddots & \ddots & \\ & & & & 1 & & & & -1 & -1 \\ \hline & & & & & 4 & & & & \\ & & & & & & 4 & & & \\ & & 0 & & & & & \ddots & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 4 \end{array}\right] \begin{bmatrix} \xi_{11} \\ \xi_{22} \\ \vdots \\ \xi_{nn} \\ \xi_{12} \\ \xi_{23} \\ \vdots \\ \xi_{n1} \end{bmatrix} = \mathbb{B}[\xi_{ij}].$$

Serendipity Theorem



$$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\lambda_a\lambda_b\} \xrightarrow{\mathbb{A}} \{\xi_{ij}\} \xrightarrow{\mathbb{B}} \{\psi_{ij}\}$$

Theorem

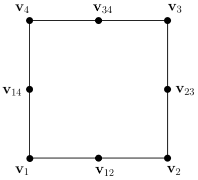
Given bounds on polygon aspect ratio (**G1**), minimum edge length (**G2**), and maximum interior angles (**G3**):

- $||\mathbb{A}||$ is uniformly bounded,
- $||\mathbb{B}||$ is uniformly bounded, and
- span$\{\psi_{ij}\} \supset \mathcal{P}_2(\mathbb{R}^2)$ = quadratic polynomials in x and y

We obtain the quadratic *a priori* error estimate: $\boxed{||u - u_h||_{H^1(\Omega)} \leq C\, h^{\mathbf{2}}\, |u|_{H^3(\Omega)}}$

RAND, G, BAJAJ *Quadratic Serendipity Finite Element on Polygons Using Generalized Barycentric Coordinates*, Math. Comp., 2011

Special case of a square



Bilinear functions are barycentric coordinates:

$$\begin{aligned}\lambda_1 &= (1-x)(1-y)\\ \lambda_2 &= x(1-y)\\ \lambda_3 &= xy\\ \lambda_4 &= (1-x)y\end{aligned}$$

Compute $[\xi_{ij}] := \left[\ \mathbb{I}\ \middle|\ \mathbb{A}'\ \right][\lambda_a\lambda_b]$

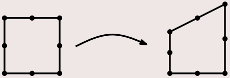
$$\begin{bmatrix}\xi_{11}\\ \xi_{22}\\ \xi_{33}\\ \xi_{44}\\ \xi_{12}\\ \xi_{23}\\ \xi_{34}\\ \xi_{14}\end{bmatrix} = \begin{bmatrix}1 & & 0 & -1 & 0\\ 0 & & 0 & 0 & -1\\ 0 & & 0 & -1 & 0\\ 0 & \cdots & 0 & 0 & -1\\ 0 & \cdots & 0 & 1/2 & 1/2\\ 0 & & 0 & 1/2 & 1/2\\ 0 & & 0 & 1/2 & 1/2\\ 0 & & 1 & 1/2 & 1/2\end{bmatrix}\begin{bmatrix}\lambda_1\lambda_1\\ \lambda_2\lambda_2\\ \lambda_3\lambda_3\\ \lambda_4\lambda_4\\ \lambda_1\lambda_2\\ \lambda_2\lambda_3\\ \lambda_3\lambda_4\\ \lambda_1\lambda_4\end{bmatrix} = \begin{bmatrix}(1-x)(1-y)(1-x-y)\\ x(1-y)(x-y)\\ xy(-1+x+y)\\ (1-x)y(y-x)\\ (1-x)x(1-y)\\ x(1-y)y\\ (1-x)xy\\ (1-x)(1-y)y\end{bmatrix}$$

$$\mathrm{span}\left\{\xi_{ii}, \xi_{i(i+1)}\right\} = \mathrm{span}\left\{1, x, y, x^2, y^2, xy, x^2y, xy^2\right\} =: \mathcal{S}_2(I^2)$$

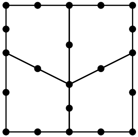
Hence, this provides a computational basis for the serendipity space $\mathcal{S}_2(I^2)$ defined in

ARNOLD, AWANOU *The serendipity family of finite elements*, Found. Comp. Math, 2011.

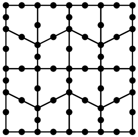
Numerical evidence for non-affine image of a square

Instead of mapping  ,
use quadratic serendipity **GBC** interpolation with mean value coordinates:

$$u_h = I_q u := \sum_{i=1}^{n} u(\mathbf{v}_i)\psi_{ii} + u\left(\frac{\mathbf{v}_i + \mathbf{v}_{i+1}}{2}\right)\psi_{i(i+1)}$$



$n = 2$



$n = 4$

Non-affine bilinear mapping

	$\|\|u - u_h\|\|_{L^2}$		$\|\|\nabla(u - u_h)\|\|_{L^2}$	
n	error	rate	error	rate
2	5.0e-2		6.2e-1	
4	6.7e-3	2.9	1.8e-1	1.8
8	9.7e-4	2.8	5.9e-2	1.6
16	1.6e-4	2.6	2.3e-2	1.4
32	3.3e-5	2.3	1.0e-2	1.2
64	7.4e-6	2.1	4.96e-3	1.1

Arnold, Boffi, Falk, Math. Comp., 2002

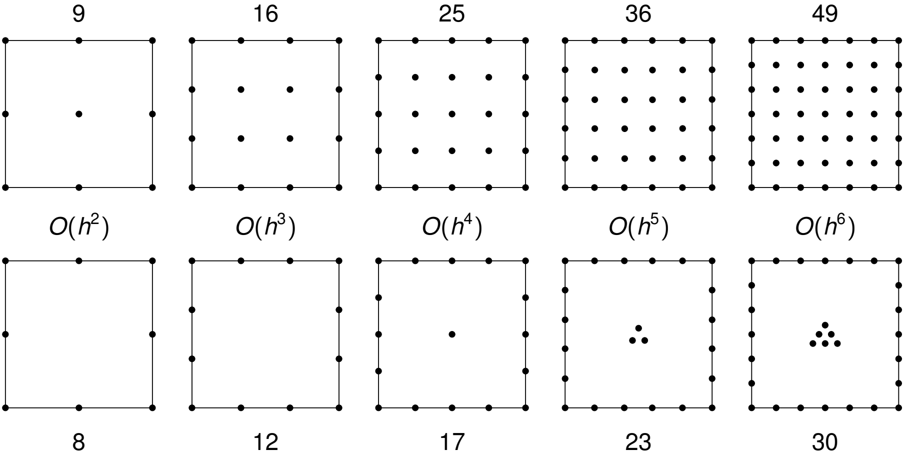
Quadratic serendipity **GBC** method

	$\|\|u - u_h\|\|_{L^2}$		$\|\|\nabla(u - u_h)\|\|_{L^2}$	
n	error	rate	error	rate
2	2.34e-3		2.22e-2	
4	3.03e-4	2.95	6.10e-3	1.87
8	3.87e-5	2.97	1.59e-3	1.94
16	4.88e-6	2.99	4.04e-4	1.97
32	6.13e-7	3.00	1.02e-4	1.99
64	7.67e-8	3.00	2.56e-5	1.99
128	9.59e-9	3.00	6.40e-6	2.00
256	1.20e-9	3.00	1.64e-6	1.96

Outline

1. Linear elements on polygons
2. Quadratic serendipity elements on polygons
3. Order r serendipity elements on n-cubes
4. Future directions

Serendipity elements on squares



For $r \geq 4$: $O(h^r)$ tensor product: $r^2 + 2r + 1$ degrees of freedom

$O(h^r)$ serendipity: $\frac{1}{2}(r^2 + 3r + 6)$ degrees of freedom

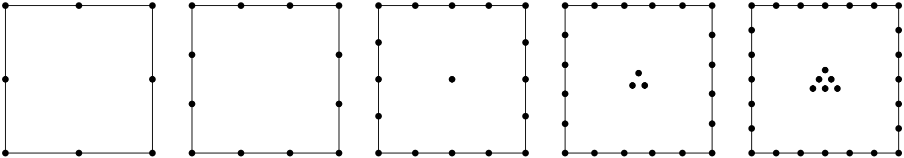
$$\underbrace{||u - u_h||_{H^1(\Omega)}}_{\text{approximation error}} \leq \underbrace{C\, h^r\, |u|_{H^{r+1}(\Omega)}}_{\text{optimal error bound}}, \quad \forall u \in H^{r+1}(\Omega).$$

Mathematical challenges

$\rightarrow$ Basis functions must be constructed to implement serendipity elements.
$\rightarrow$ Current constructions lack key mathematical properties, limiting their broader usage

Goal: Construct basis functions for serendipity elements satisfying the following:

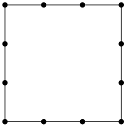
- **Symmetry:** Accommodate interior degrees of freedom that grow according to triangular numbers on square-shaped elements.
- **Tensor product structure:** Write as linear combinations of standard tensor product functions.
- **Hierarchical:** Generalize to methods on *n*-cubes for any $n \geq 2$, allowing restrictions to lower-dimensional faces.



$O(h^2)$	$O(h^3)$	$O(h^4)$	$O(h^5)$	$O(h^6)$
8	12	17	23	30

Which monomials do we need?

$O(h^3)$ serendipity element:



$$\{\underbrace{\overbrace{1, x, y, x^2, y^2, xy, x^3, y^3, x^2y, xy^2}^{\substack{\text{total degree at most cubic} \\ \text{(req. for } O(h^3) \text{ approximation)}}}, x^3y, xy^3, x^2y^2, x^3y^2, x^2y^3, x^3y^3}_{\substack{\text{at most cubic in each variable} \\ \text{(used in } O(h^3) \text{ tensor product methods)}}}\}$$

We need an intermediate set of 12 monomials!

The **superlinear** degree of a polynomial ignores linearly-appearing variables.

Example: $\text{sldeg}(xy^3) = 3$, even though $\deg(xy^3) = 4$

Definition: $\text{sldeg}(x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}) := \left(\sum_{i=1}^{n} e_i\right) - \#\{e_i \;:\; e_i = 1\}$

$$\{\underbrace{1, x, y, x^2, y^2, xy, x^3, y^3, x^2y, xy^2, x^3y, xy^3}_{\text{superlinear degree at most 3 (dim=12)}}, x^2y^2, x^3y^2, x^2y^3, x^3y^3\}$$

Arnold, Awanou *The serendipity family of finite elements*, Found. Comp. Math, 2011.

Superlinear polynomials form a lower set

Given a monomial $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$,

associate the multi-index of d non-negative integers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}_0^d$.

Define the superlinear norm of α as $$|\alpha|_{sprlin} := \sum_{\substack{j=1 \\ \alpha_j \geq 2}}^{d} \alpha_j,$$

so that the superlinear multi indices are $$S_r = \left\{ \alpha \in \mathbb{N}_0^d : |\alpha|_{sprlin} \leq r \right\}.$$

Observe that S_r has a partial ordering $\mu \leq \alpha$ means $\mu_i \leq \alpha_i$.

Thus S_r is a **lower set**, meaning $$\boxed{\alpha \in S_r,\, \mu \leq \alpha \implies \mu \in S_r}$$

We can thus apply the following recent result.

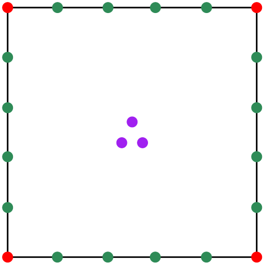
Theorem (Dyn and Floater, 2013)

Fix a lower set $L \subset \mathbb{N}_0^d$ and points $z_\alpha \in \mathbb{R}^d$ for all $\alpha \in L$. For any sufficiently smooth d-variate real function f, there is a unique polynomial $p \in \mathrm{span}\{x^\alpha : \alpha \in L\}$ that interpolates f at the points z_α, with partial derivative interpolation for repeated z_α values.

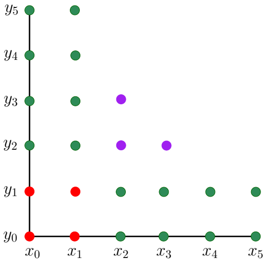
DYN AND FLOATER *Multivariate polynomial interpolation on lower sets*, J. Approx. Th., to appear.

Partitioning and reordering the multi-indices

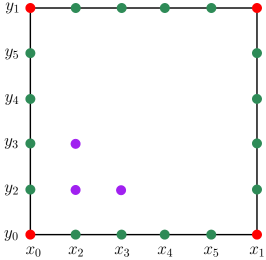
By a judicious choice of the interpolation points $z_\alpha = (x_i, y_j)$, we recover the dimensionality associations of the degrees of freedom of serendipity elements.



The order 5 serendipity element, with degrees of freedom color-coded by dimensionality.



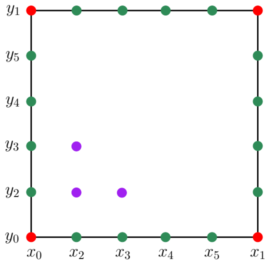
The lower set S_5, with equivalent color coding.



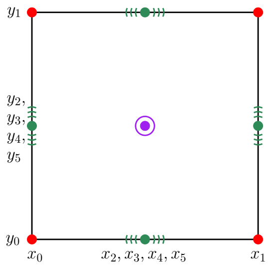
The lower set S_5, with domain points z_α reordered.

Symmetrizing the multi-indices

By collecting the re-ordered interpolation points $z_\alpha = (x_i, y_j)$, at midpoints of the associated face, we recover the dimensionality associations of the degrees of freedom of serendipity elements.



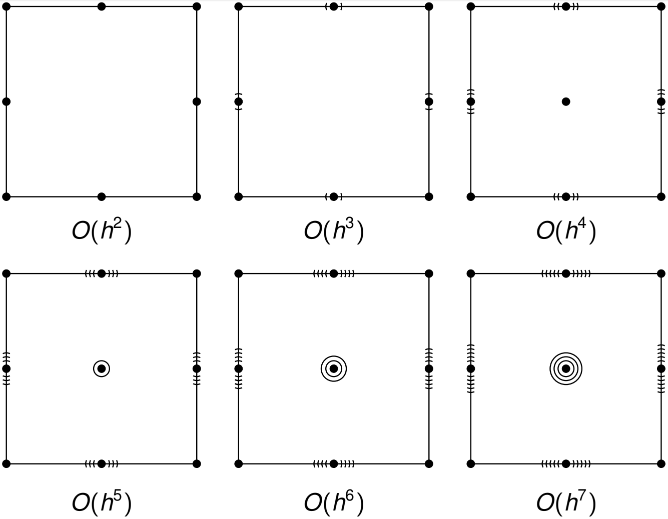
The lower set S_5, with domain points z_α reordered.



A symmetric reordering, with multiplicity. The associated interpolant recovers values at dots, three partial derivatives at edge midpoints, and two partial derivatives at the face midpoint.

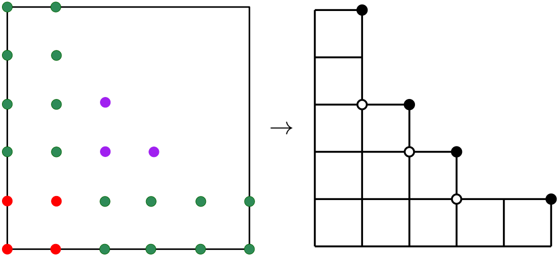
2D symmetric serendipity elements

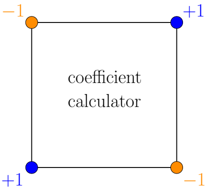
Symmetry: Accommodate interior degrees of freedom that grow according to triangular numbers on square-shaped elements.



Tensor product structure

The Dyn-Floater interpolation scheme is expressed in terms of tensor product interpolation over 'maximal blocks' in the set using an inclusion-exclusion formula.





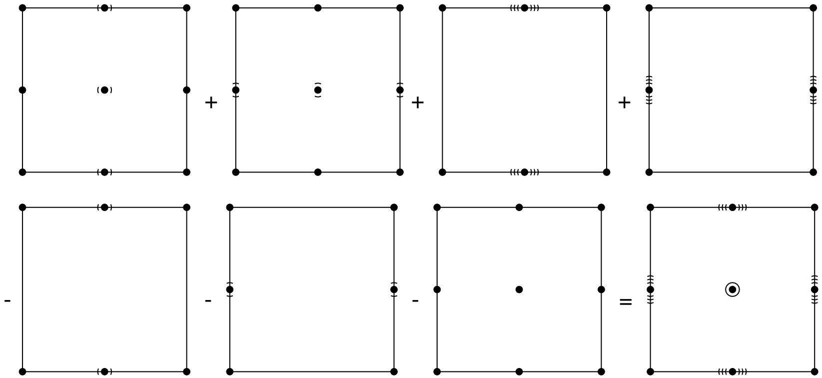
Put differently, the linear combination is the sum over *all* blocks within the lower set with coefficients determined as follows:

$\rightarrow$ Place the coefficient calculator at the extremal block corner.
$\rightarrow$ Add up all values appearing in the lower set.
$\rightarrow$ The coefficient for the block is the value of the sum.

Hence: black dots $\rightarrow$ +1; white dots $\rightarrow$ -1; others $\rightarrow$ 0.

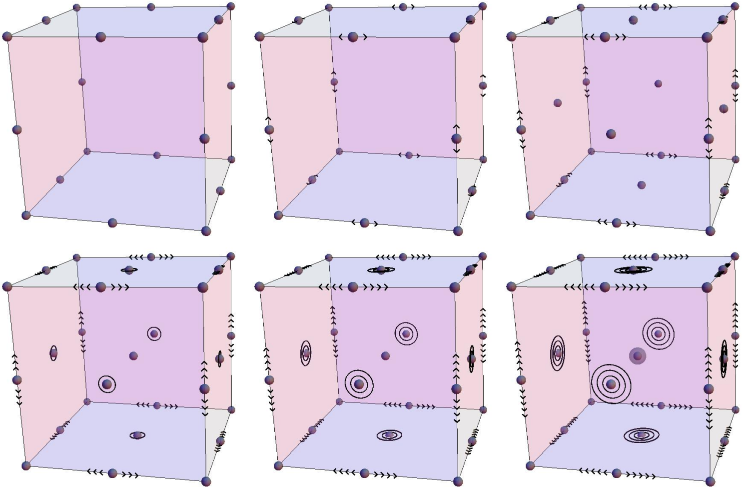
Linear combination of tensor products

Tensor product structure: Write basis functions as linear combinations of standard tensor product functions.



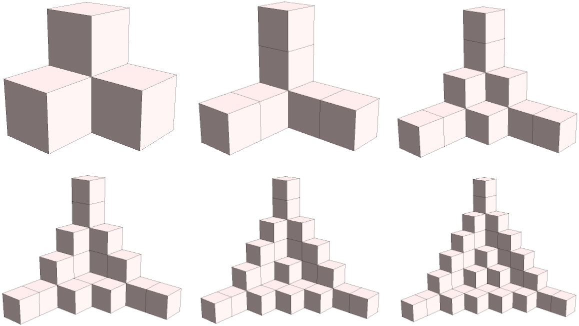
3D elements

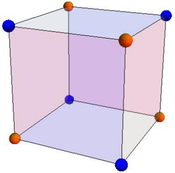
Hierarchical: Generalize to methods on *n*-cubes for any $n \geq 2$, allowing restrictions to lower-dimensional faces.



3D coefficient computation

Lower sets for superlinear polynomials in 3 variables:





Decomposition into a linear combination of tensor product interpolants works the same as in 2D, using the 3D coefficient calculator at left. (Blue $\rightarrow$ +1; Orange $\rightarrow$ -1).

FLOATER, G, *Nodal bases for the serendipity family of finite elements*, Submitted, 2014. Available as arXiv:1404.6275

Outline

Future Directions

- Expand serendipity results from cubes to polyhedra.
- Incorporate elements into finite element software packages.
- Analyze speed vs. accuracy trade-offs.

	1	2	3	4	5	6	7	$r \geq 2n$
$n = 2$								
$\dim \mathcal{Q}_r$	4	9	16	25	36	49	64	$r^2 + 2r + 1$
$\dim \mathcal{S}_r$	4	8	12	17	23	30	38	$\frac{1}{2}(r^2 + 3r + 6)$
$n = 3$								
$\dim \mathcal{Q}_r$	8	27	64	125	216	343	512	$r^3 + 3r^2 + 3r + 1$
$\dim \mathcal{S}_r$	8	20	32	50	74	105	144	$\frac{1}{6}(r^3 + 6r^2 + 29r + 24)$

Acknowledgments

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Chandrajit Bajaj	UT Austin
Alexander Rand	UT Austin / CD-adapco
Michael Holst	UC San Diego
National Biomedical Computation Resource	UC San Diego
Michael Floater	University of Oslo
N. Sukumar	UC Davis

Thanks for the invitation to speak!

Slides and pre-prints: `http://math.arizona.edu/~agillette/`