The Geometry of Interpolation Error Estimates for Isogeometric Analysis

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Motivation

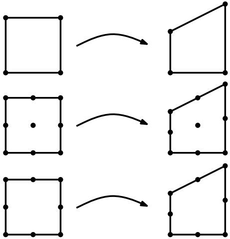
Suppose we want to solve a finite element problem with

geometry map $\mathbf{T}$: reference element $\longrightarrow$ physical elements

interpolation map $\mathcal{I}$: physical DoFs $\longrightarrow$ interpolated function

If $\mathbf{T}$ is non-affine, *a priori* error estimates for $\mathcal{I}$ can converge at sub-optimal rates.

ARNOLD, BOFFI, FALK *Approximation by Quadrilateral Finite Elements*, Math. Comp., 2002



Lagrange: $||u - \mathcal{I}_\ell u||_{H^1(\Omega)} \leq c\, h\, |u|_{H^2(\Omega)}, \quad \forall u \in H^2(\Omega)$

quadratic: $||u - \mathcal{I}_q u||_{H^1(\Omega)} \leq c\, h^2\, |u|_{H^3(\Omega)}, \quad \forall u \in H^3(\Omega)$

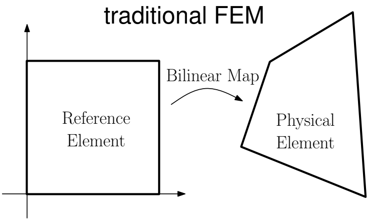
serendipity: $||u - \mathcal{I}_s u||_{H^1(\Omega)} \leq c\, h\, |u|_{H^3(\Omega)}, \quad \forall u \in H^3(\Omega)$

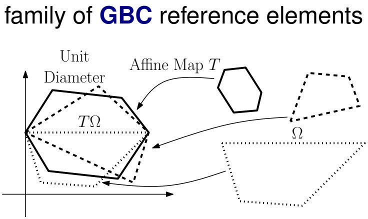
How can the usual $O(h^2)$ rate for the serendipity element be recovered for non-affine $\mathbf{T}$?

Isogeometric solution: Define $\mathcal{I}$ on physical elements; characterize geometric dependence.

Approach

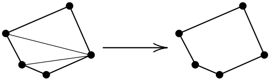
Use **G**eneralized **B**arycentric **C**oordinates **(GBCs)** to create polygonal finite elements.



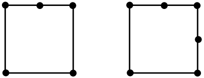


Advantages of the approach

- Builds on rich theory of **GBCs** from graphics literature, amenable to IGA.
- Some **GBC** elements are not sensitive to large angles, allowing remeshing:



- Provides canonical adaptive meshing elements:



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Definition

Let Ω be a convex polygon in $\mathbb{R}^2$ with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Functions $\lambda_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, n$ are called **barycentric coordinates** on Ω if they satisfy two properties:

1. **Non-negative**: $\lambda_i \geq 0$ on Ω.
2. **Linear Completeness**: For any linear function $L : \Omega \to \mathbb{R}$, $L = \sum_{i=1}^{n} L(\mathbf{v}_i)\lambda_i$.

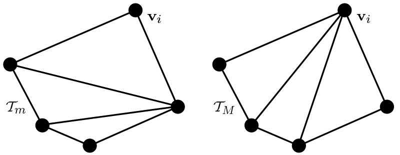
Any set of barycentric coordinates under this definition also satisfies:

3. **Partition of unity:** $\sum_{i=1}^{n} \lambda_i \equiv 1$.
4. **Linear precision:** $\sum_{i=1}^{n} \mathbf{v}_i \lambda_i(\mathbf{x}) = \mathbf{x}$.
5. **Interpolation:** $\lambda_i(\mathbf{v}_j) = \delta_{ij}$.

Theorem [Warren, 2003]

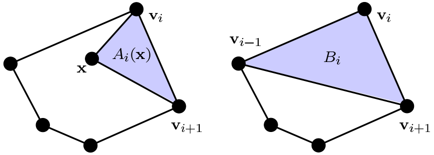
If the λ_i are rational functions of degree $n-2$, then they are unique.

Many generalizations to choose from . . .

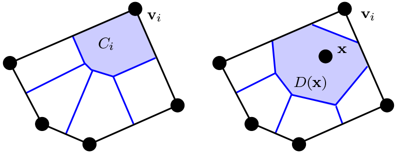


- Triangulation
  ⇒ FLOATER, HORMANN, KÓS, *A general construction of barycentric coordinates over convex polygons*, 2006

$$0 \leq \lambda_i^{\mathcal{T}_m}(\mathbf{x}) \leq \lambda_i(\mathbf{x}) \leq \lambda_i^{\mathcal{T}_M}(\mathbf{x}) \leq 1$$

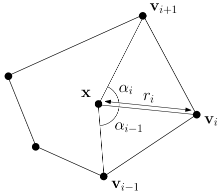


- Wachspress
  ⇒ WACHSPRESS, *A Rational Finite Element Basis*, 1975.

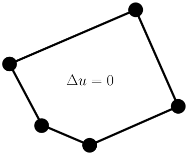


- Sibson / Laplace
  ⇒ SIBSON, *A vector identity for the Dirichlet tessellation*, 1980.
  ⇒ HIYOSHI, SUGIHARA, *Voronoi-based interpolation with higher continuity*, 2000.

Many generalizations to choose from . . .



- Mean value
  ⇒ FLOATER, *Mean value coordinates*, 2003.
  ⇒ FLOATER, KÓS, REIMERS, *Mean value coordinates in 3D*, 2005.

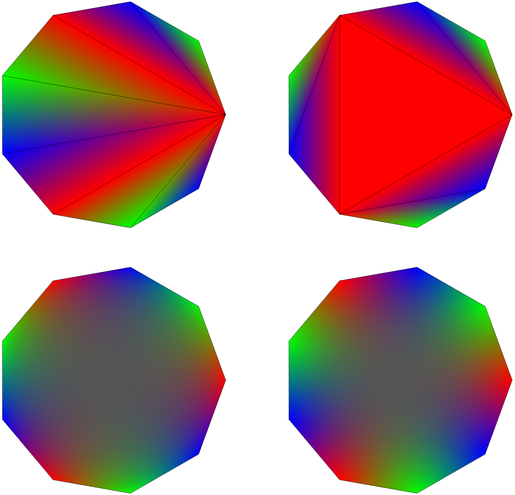


- Harmonic
  ⇒ WARREN, *Barycentric coordinates for convex polytopes*, 1996.
  ⇒ WARREN, SCHAEFER, HIRANI, DESBRUN, *Barycentric coordinates for convex sets*, 2007.

Many more in graphics contexts...

Comparison via 'eyeball' norm

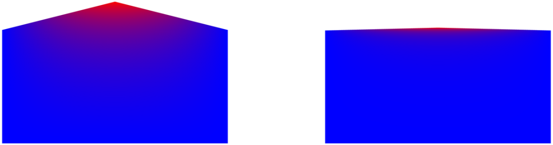
Triangulated | Triangulated



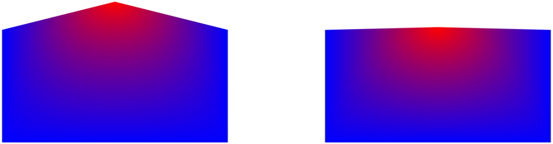
Wachspress | Mean Value

Comparison via 'eyeball' norm

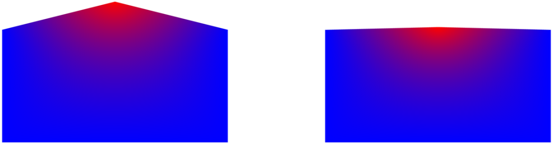
Wachspress



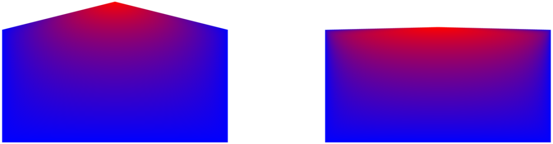
Sibson



Mean Value



Discrete Harmonic



Outline

Optimal Convergence Estimates on Polygons

Let Ω be a convex polygon with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

For linear elements, an **optimal convergence estimate** has the form

$$\underbrace{\left|\left| u - \sum_{i=1}^{n} u(\mathbf{v}_i)\lambda_i \right|\right|_{H^1(\Omega)}}_{\text{approximation error}} \leq \underbrace{C\,\mathrm{diam}(\Omega)\,|u|_{H^2(\Omega)}}_{\text{optimal error bound}}, \quad \forall u \in H^2(\Omega). \tag{1}$$

The **Bramble-Hilbert lemma** in this context says that any $u \in H^2(\Omega)$ is close to a first order polynomial in H^1 norm.

VERFÜRTH, *A note on polynomial approximation in Sobolev spaces*, Math. Mod. Num. An., 2008.
DEKEL, LEVIATAN, *The Bramble-Hilbert lemma for convex domains*, SIAM J. Math. An., 2004.

For (1), it suffices to prove an **H^1-interpolant estimate** over domains of diameter one:

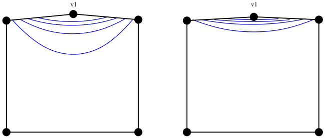
$$\left|\left| \sum_{i=1}^{n} u(\mathbf{v}_i)\lambda_i \right|\right|_{H^1(\Omega)} \leq C_I\,||u||_{H^2(\Omega)}, \quad \forall u \in H^1(\Omega). \tag{2}$$

For (2), it suffices to **bound the gradients** of the $\{\lambda_i\}$, i.e. prove $\exists\ C_\lambda \in \mathbb{R}$ such that

$$||\nabla\lambda_i||_{L^2(\Omega)} \leq C_\lambda. \tag{3}$$

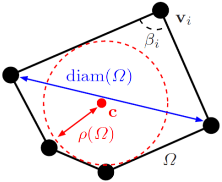
Geometric Hypotheses for Convergence Estimates

To bound the gradients of the coordinates, we need control of the element geometry.



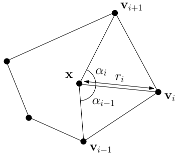
Let $\rho(\Omega)$ denote the radius of the largest inscribed circle.
The **aspect ratio** γ is defined by

$$\gamma = \frac{\text{diam}(\Omega)}{\rho(\Omega)} \in (2, \infty)$$



Three possible geometric conditions on a polygonal mesh:

G1.	BOUNDED ASPECT RATIO:	$\exists\ \gamma^* < \infty$	such that	$\gamma < \gamma^*$
G2.	MINIMUM EDGE LENGTH:	$\exists\ d_* > 0$	such that	$\lvert\mathbf{v}_i - \mathbf{v}_{i-1}\rvert > d_*$
G3.	MAXIMUM INTERIOR ANGLE:	$\exists\ \beta^* < \pi$	such that	$\beta_i < \beta^*$

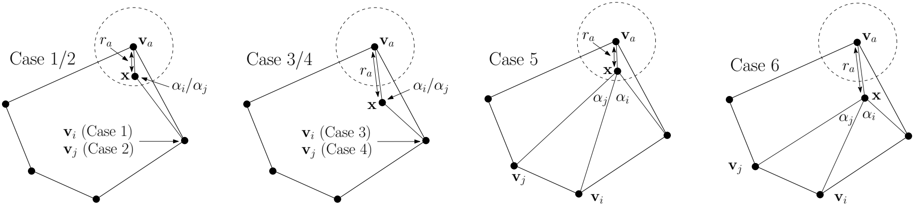


$$\lambda_i^{\mathrm{MV}}(\mathbf{x}) := \frac{w_i(\mathbf{x})}{\sum_{j=1}^n w_j(\mathbf{x})} \qquad w_i(\mathbf{x}) := \frac{\tan\left(\frac{\alpha_i(\mathbf{x})}{2}\right) + \tan\left(\frac{\alpha_{i-1}(\mathbf{x})}{2}\right)}{||\mathbf{v}_i - \mathbf{x}||}$$

Main Problem: ∇w_i is complicated. Just part of the expression is

$$\nabla\left[\frac{\tan\left(\frac{\alpha_i(\mathbf{x})}{2}\right)}{r_i(\mathbf{x})}\right] = \frac{\sec^2\left(\frac{\alpha_i(\mathbf{x})}{2}\right)\nabla\alpha_i/2}{r_i(\mathbf{x})} - \frac{\tan\left(\frac{\alpha_i(\mathbf{x})}{2}\right)}{(r_i(\mathbf{x}))^2}\nabla r_i(\mathbf{x})$$

Solution: Divide analysis into six cases based on proximity to $\mathbf{v}_a$ and size of α_i and α_j



RAND, GILLETTE, BAJAJ *Interpolation Error Estimates for Mean Value Coordinates*, submitted

Polygonal Finite Element Optimal Convergence

Theorem

In the table, any necessary geometric criteria to achieve the H^1 interpolant estimate are denoted by N. The set of geometric criteria denoted by S in each row ***taken together*** are sufficient to guarantee the H^1 interpolant estimate.

		G1 (aspect ratio)	**G2** (min edge length)	**G3** (max interior angle)
Triangulated	λ^{Tri}	-	-	S,N
Wachspress	λ^{Wach}	S	S	S,N
Sibson	λ^{Sibs}	S	S	-
Mean Value	λ^{MV}	S	S	-
Harmonic	λ^{Har}	S	-	-

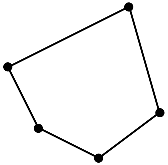
GILLETTE, RAND, BAJAJ *Error Estimates for Generalized Barycentric Interpolation*
Advances in Computational Mathematics, to appear, 2011

RAND, GILLETTE, BAJAJ *Interpolation Error Estimates for Mean Value Coordinates*, submitted

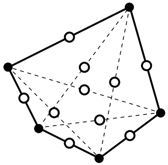
Outline

From linear to quadratic elements

A naïve quadratic element is formed by products of linear element basis functions:



$$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\lambda_a\lambda_b\}$$



Why is this naïve?

- For an n-gon, this construction gives $n + \binom{n}{2}$ basis functions $\lambda_a\lambda_b$
- The space of quadratic polynomials is only dimension 6: $\{1, x, y, xy, x^2, y^2\}$
- Conforming to a linear function on the boundary requires 2 degrees of freedom per edge $\Rightarrow$ *only* $2n$ *functions needed!*

Problem Statement

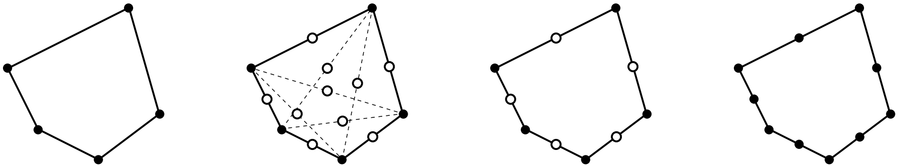
Construct $2n$ basis functions associated to the vertices and edge midpoints of an arbitrary n-gon such that a quadratic convergence estimate is obtained.

Polygonal Quadratic Serendipity Elements

We define matrices $\mathbb{A}$ and $\mathbb{B}$ to reduce the naïve quadratic basis.

filled dot $=$ Lagrangian domain point
$=$ all functions in the set evaluate to 0
except the associated function which evaluates to 1

open dot $=$ non-Lagrangian domain point
$=$ partition of unity satisfied, but not Lagrange property



$$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\lambda_a\lambda_b\} \xrightarrow{\mathbb{A}} \{\xi_{ij}\} \xrightarrow{\mathbb{B}} \{\psi_{ij}\}$$

Linear	Quadratic	Serendipity	Lagrange
n	$n+\binom{n}{2}$	$2n$	$2n$

From quadratic to serendipity

The bases are ordered as follows:

ξ_{ii} and $\lambda_a\lambda_a$ = basis functions associated with vertices
$\xi_{i(i+1)}$ and $\lambda_a\lambda_{a+1}$ = basis functions associated with edge midpoints
$\lambda_a\lambda_b$ = basis functions associated with interior diagonals, i.e. $b \notin \{a-1, a, a+1\}$

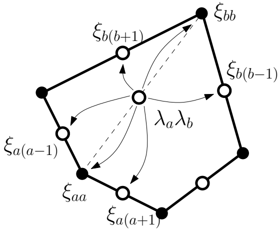
Serendipity basis functions ξ_{ij} are a linear combination of pairwise product functions $\lambda_a\lambda_b$:

$$\begin{bmatrix} \xi_{ii} \\ \vdots \\ \xi_{i(i+1)} \end{bmatrix} = \mathbb{A} \begin{bmatrix} \lambda_a\lambda_a \\ \vdots \\ \lambda_a\lambda_{a+1} \\ \vdots \\ \lambda_a\lambda_b \end{bmatrix} = \begin{bmatrix} c_{11}^{11} & \cdots & c_{ab}^{11} & \cdots & c_{(n-2)n}^{11} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{11}^{ij} & \cdots & c_{ab}^{ij} & \cdots & c_{(n-2)n}^{ij} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{11}^{n(n+1)} & \cdots & c_{ab}^{n(n+1)} & \cdots & c_{(n-2)n}^{n(n+1)} \end{bmatrix} \begin{bmatrix} \lambda_a\lambda_a \\ \vdots \\ \lambda_a\lambda_{a+1} \\ \vdots \\ \lambda_a\lambda_b \end{bmatrix}$$

From quadratic to serendipity

We ***require*** the serendipity basis to have quadratic approximation power:

Constant precision:	1	$=$	$\sum_i \xi_{ii} + 2\xi_{i(i+1)}$
Linear precision:	$\mathbf{x}$	$=$	$\sum_i \mathbf{v}_i \xi_{ii} + 2\mathbf{v}_{i(i+1)}\xi_{i(i+1)}$
Quadratic precision:	$\mathbf{x}\mathbf{x}^T$	$=$	$\sum_i \mathbf{v}_i\mathbf{v}_i^T \xi_{ii} + (\mathbf{v}_i\mathbf{v}_{i+1}^T + \mathbf{v}_{i+1}\mathbf{v}_i^T)\xi_{i(i+1)}$

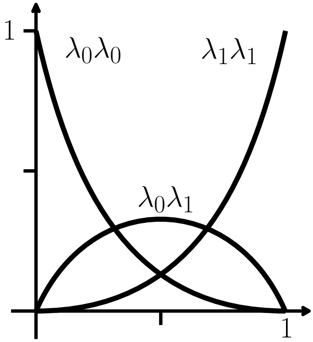


Theorem

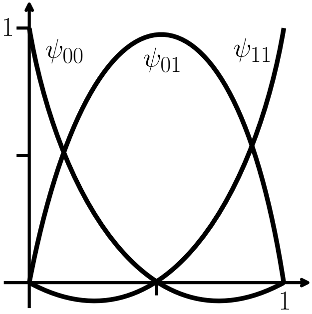
Constants $\{c_{ij}^{ab}\}$ exist for **any** convex polygon such that the resulting basis $\{\xi_{ij}\}$ satisfies constant, linear, and quadratic precision requirements.

Pairwise products vs. Lagrange basis

Even in 1D, pairwise products of barycentric functions do not form a Lagrange basis at interior degrees of freedom:



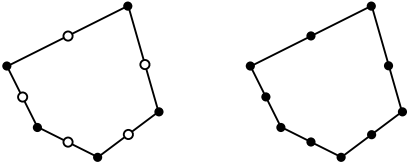
Pairwise products



Lagrange basis

Translation between these two bases is straightforward and generalizes to the higher dimensional case...

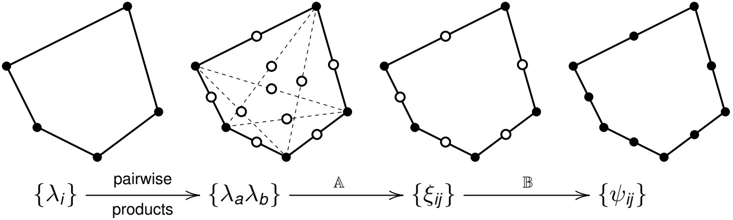
From serendipity to Lagrange



$$\{\xi_{ij}\} \xrightarrow{\quad\quad\quad\mathbb{B}\quad\quad\quad} \{\psi_{ij}\}$$

$$[\psi_{ij}] = \begin{bmatrix} \psi_{11} \\ \psi_{22} \\ \vdots \\ \psi_{nn} \\ \psi_{12} \\ \psi_{23} \\ \vdots \\ \psi_{n1} \end{bmatrix} = \left[\begin{array}{ccccc|ccccc} 1 & & & & & -1 & & \cdots & & -1 \\ & 1 & & & & -1 & -1 & \cdots & & \\ & & \ddots & & & & \ddots & \ddots & & \\ & & & \ddots & & & & \ddots & \ddots & \\ & & & & 1 & & & & -1 & -1 \\ \hline & & & & & 4 & & & & \\ & & & & & & 4 & & & \\ & & 0 & & & & & \ddots & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 4 \end{array}\right] \begin{bmatrix} \xi_{11} \\ \xi_{22} \\ \vdots \\ \xi_{nn} \\ \xi_{12} \\ \xi_{23} \\ \vdots \\ \xi_{n1} \end{bmatrix} = \mathbb{B}[\xi_{ij}].$$

Serendipity Theorem



Theorem

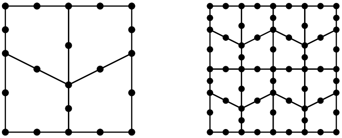
Given bounds on polygon aspect ratio (G1), minimum edge length (G2), and maximum interior angles (G3):

- $||\mathbb{A}||$ is uniformly bounded,
- $||\mathbb{B}||$ is uniformly bounded, and
- $\text{span}\{\psi_{ij}\} \supset \mathcal{P}_2(\mathbb{R}^2)$ = quadratic polynomials in x and y

RAND, GILLETTE, BAJAJ *Quadratic Serendipity Finite Element on Polygons Using Generalized Barycentric Coordinates*, Submitted*, 2011

* Revised version at my website; will be posted to the arXiV soon.

Numerical evidence



Using quadratic serendipity **GBC** interpolation with mean value coordinates:

$$u_h = I_q u := \sum_{i=1}^{n} u(\mathbf{v}_i)\psi_{ii} + u\left(\frac{\mathbf{v}_i + \mathbf{v}_{i+1}}{2}\right)\psi_{i(i+1)}$$

Non-affine bilinear mapping

	$\|\|u - u_h\|\|_{L^2}$		$\|\|\nabla(u - u_h)\|\|_{L^2}$	
n	error	rate	error	rate
2	5.0e-2		6.2e-1	
4	6.7e-3	2.9	1.8e-1	1.8
8	9.7e-4	2.8	5.9e-2	1.6
16	1.6e-4	2.6	2.3e-2	1.4
32	3.3e-5	2.3	1.0e-2	1.2
64	7.4e-6	2.1	4.96e-3	1.1

Arnold, Boffi, Falk, Math. Comp., 2002

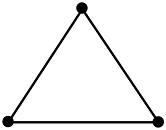
Quadratic serendipity **GBC** method

	$\|\|u - u_h\|\|_{L^2}$		$\|\|\nabla(u - u_h)\|\|_{L^2}$	
n	error	rate	error	rate
2	2.34e-3		2.22e-2	
4	3.03e-4	2.95	6.10e-3	1.87
8	3.87e-5	2.97	1.59e-3	1.94
16	4.88e-6	2.99	4.04e-4	1.97
32	6.13e-7	3.00	1.02e-4	1.99
64	7.67e-8	3.00	2.56e-5	1.99
128	9.59e-9	3.00	6.40e-6	2.00
256	1.20e-9	3.00	1.64e-6	1.96

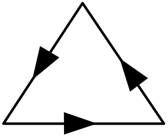
Outline

From scalar to vector elements

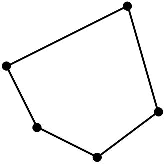
Barycentric functions are used to define H(curl) vector elements on triangles:



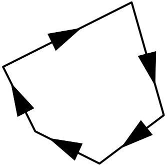
$$\{\lambda_i\} \xrightarrow[\text{construction}]{\text{Whitney}} \{\lambda_a \nabla \lambda_b - \lambda_b \nabla \lambda_a\}$$



Generalized barycentric functions provide H(curl) elements on polygons:



$$\{\lambda_i\} \xrightarrow[\text{construction}]{\text{Whitney}} \{\lambda_a \nabla \lambda_b - \lambda_b \nabla \lambda_a\}$$

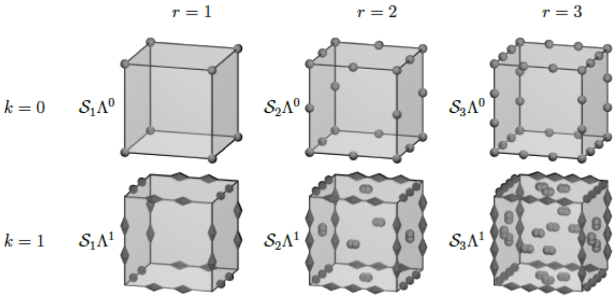


This idea fits naturally into the framework of **Discrete Exterior Calculus** and suggests a wide range of applications. . . *. . . work in progress*

GILLETTE, BAJAJ *Dual Formulations of Mixed Finite Element Methods with Applications*
Computer-Aided Design 43:10, pages 1213-1221, 2011.

Basis functions for serendipity spaces

Recent work characterized serendipity spaces in *n* dimensions for scalar fields, vector fields, and their generalization to differential *k*-forms:



ARNOLD, AWANOU *Finite Element Differential Forms on Cubical Meshes*
arXiv:1204.2595, 2012.

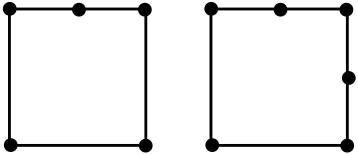
The results of that paper coupled with the techniques used here suggest a variety of new research directions. . . *. . . work in progress*

GBCs for adaptive algorithms

Recall from the table of geometric dependencies for gradient bounds:

		G1 (aspect ratio)	**G2** (min edge length)	**G3** (max interior angle)
Mean Value	λ^{MV}	S	S	-

Thus, the quadratic serendipity construction with mean value coordinates can still allow quadratic convergence for elements with interior angles of π, in particular, for those used in adaptive methods:



This could help with isogeometric analysis of adaptive methods or T-splines.

References

- Kai Hormann's webpage on generalized barycentric coordinates:

  http://www.inf.usi.ch/hormann/barycentric

- NSF workshop on generalized barycentric coordinates in NYC (July 25-27):

  http://www.inf.usi.ch/hormann/nsfworkshop

- Slides and pre-prints are available at:

  http://ccom.ucsd.edu/∼agillette