How to speed up your tensor product finite element code without really trying

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joint work with
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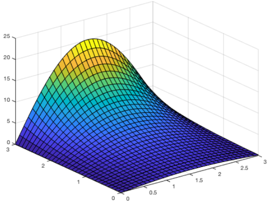
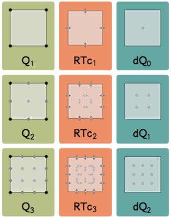
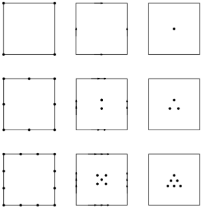
Victoria Sanders, University of Arizona







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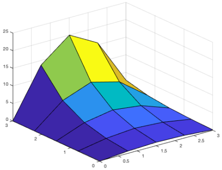
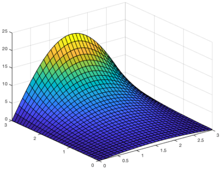
The original “serendipity phenomenon”



$\ell = 4$



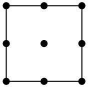
$\ell = 32$

Finite element method for $\Delta u = 0$.

Boundary data: $\sin(x)\ e^y$

Domain: $[0,3]^2$, with $\ell \times \ell$ square grid.

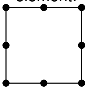
Quadratic tensor product element:



$\mathcal{Q}_2^-$

ℓ	DoFs	$\|\|u - u_h\|\|_2$	ratio	order	$\|\|\nabla u - \nabla u_h\|\|_2$	ratio	order
2	25	4.2029e-01			1.9410e+00		
4	81	5.7476e-02	7.313	2.870	5.0683e-01	3.830	1.937
8	289	7.3802e-03	7.788	2.961	1.2823e-01	3.952	1.983
16	1089	9.2909e-04	7.943	2.990	3.2157e-02	3.988	1.996
32	**4225**	1.1635e-04	7.986	**2.997**	8.0455e-03	3.997	**1.999**

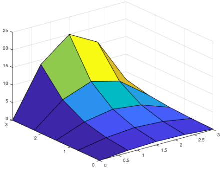
Quadratic serendipity element:



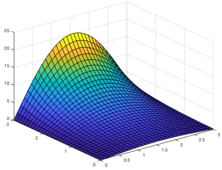
$\mathcal{S}_2^-$

ℓ	DoFs	$\|\|u - u_h\|\|_2$	ratio	order	$\|\|\nabla u - \nabla u_h\|\|_2$	ratio	order
2	21	5.6921e-01	0.000	0.000	2.4006e+00	0.000	0.000
4	65	6.0711e-02	9.376	3.229	5.3156e-01	4.516	2.175
8	225	7.4447e-03	8.155	3.028	1.2947e-01	4.106	2.038
16	833	9.3040e-04	8.002	3.000	3.2221e-02	4.018	2.007
32	**3201**	1.1637e-04	7.995	**2.999**	8.0491e-03	4.003	**2.001**

The original "serendipity" phenomenon



$\ell = 4$



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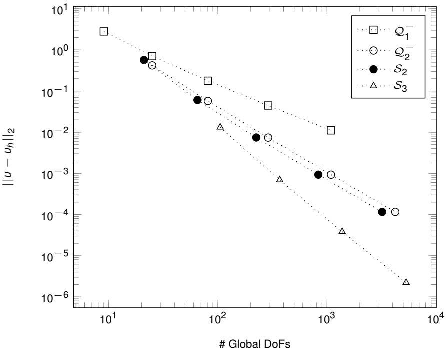
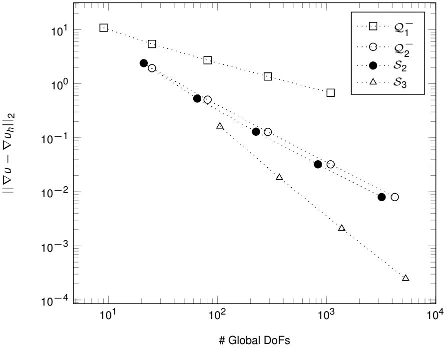
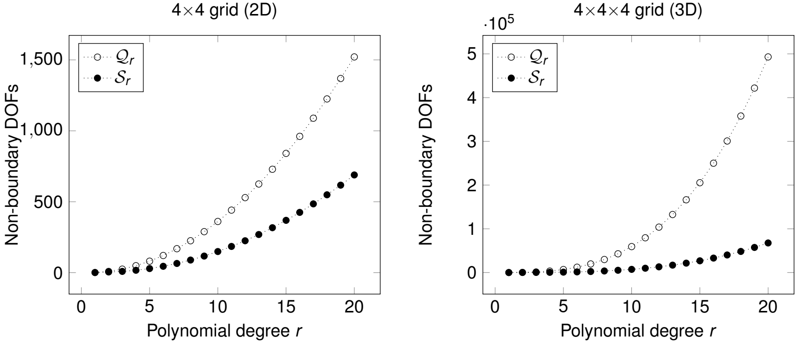
Domain: $[0, 3]^2$, with $\ell \times \ell$ square grid.





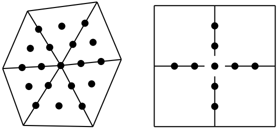
How much of a savings in DoFs can we get for large r?

Serendipity per-element DoF savings grow with r



- $\rightarrow$ DoFs per $\mathcal{Q}_r^-$ (scalar) element in dim $n = (r+1)^n$
- $\rightarrow$ DoFs per $\mathcal{S}_r$ (scalar) element in dim $n = \mathcal{O}(r^n/n!)$
- $\rightarrow$ In 2D, for large r, $\mathcal{Q}_r$ has $\approx$ 2 times as many DoFs per element as $\mathcal{S}_r$
- $\rightarrow$ **In 3D**, for large r, $\mathcal{Q}_r$ has $\approx$ **5.8 times** as many DoFs per element as $\mathcal{S}_r$, including **more than 2 times** as many DoFs shared between elements!

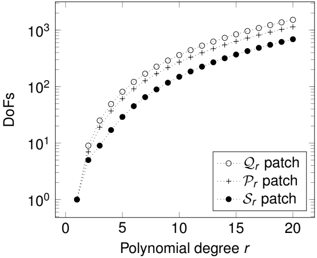
Additional potential savings for solvers



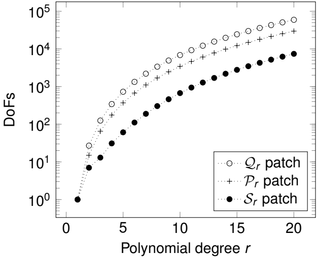
Patch-based solvers depend on a stencil of DoFs around each vertex in a mesh. Stencils for $\mathcal{P}_3$ on a triangular mesh and $\mathcal{S}_3$ on a quad mesh are shown.

$\hookrightarrow$ from a proposal with Rob Kirby (Baylor U.); currently under review

2D patches (6 triangles vs. 4 quads)



3D patches (24 tets vs. 8 hexes)



Ex: In 3D, a $\mathcal{Q}_5$ patch has $\approx$ **12 times** the number of DoFs as a $\mathcal{S}_5$ patch

$\Longrightarrow$ a quadratic order complexity solver with $\mathcal{Q}_5$ patches would have $\approx$ **144 times** longer run times than one with $\mathcal{S}_5$ patches!

Takeaway: robustly implementing serendipity elements should allow significant reduction in computational cost with no loss in order of accuracy.

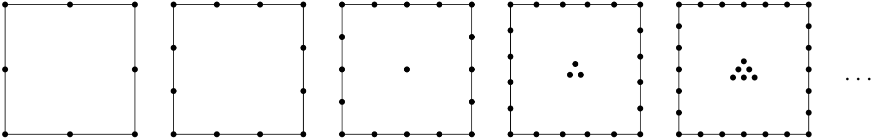
Outline

1. Serendipity methods: a review of their potential
2. Recent mathematical advances in serendipity theory
3. Three challenges facing large-scale implementation

Two key insights from Arnold and Awanou

$\to$ Scalar serendipity elements exist for any order $r \geq 1$ in any dimension $n \geq 2$.

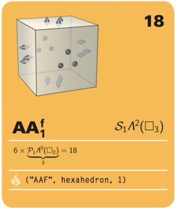
ARNOLD, AWANOU "The serendipity family of finite elements ", *Foundations of Computational Mathematics*, 2011



$r = 2$ $\quad$ $r = 3$ $\quad$ $r = 4$ $\quad$ $r = 5$ $\quad$ $r = 6$ $\quad \cdots$

$\to$ Scalar serendipity elements are part of a family of finite element differential forms.

ARNOLD, AWANOU "Finite element differential forms on cubical meshes", *Mathematics of Computation*, 2013

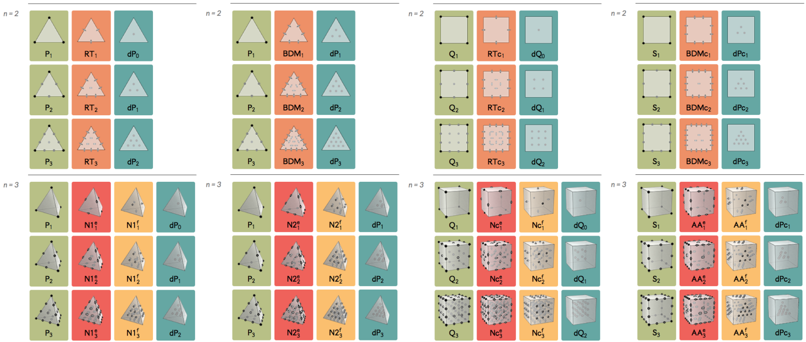


Ex: $\mathcal{S}_1\Lambda^2(\square_3)$ is an element for

$$
\begin{array}{rcl}
r = 1 & \to & \text{linear order of error decay} \\
k = 2 & \to & \text{conformity in } \Lambda^2(\mathbb{R}^3) \rightsquigarrow H(\mathrm{div}) \\
n = 3 & \to & \text{domains in } \mathbb{R}^3
\end{array}
$$

The 'Periodic Table of the Finite Elements'

ARNOLD, LOGG, "Periodic table of the finite elements," *SIAM News*, 2014.

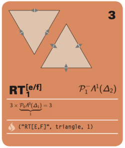


Classification of many common conforming finite element types.

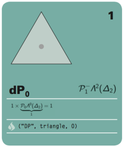
$n \rightarrow$ Domains in $\mathbb{R}^2$ (top half) and in $\mathbb{R}^3$ (bottom half)
$r \rightarrow$ Order $1, 2, 3$ of error decay (going down columns)
$k \rightarrow$ Conformity type $k = 0, \ldots, n$ (going across a row)

Geometry types: Simplices (left half) and cubes (right half).

Method selection and cochain complexes

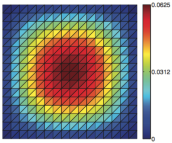


$\subset H(\text{div})$



$\subset L^2$

$\Rightarrow$



Provably stable method

converges to
$\mathbf{u} = x(1-x)y(1-y)$

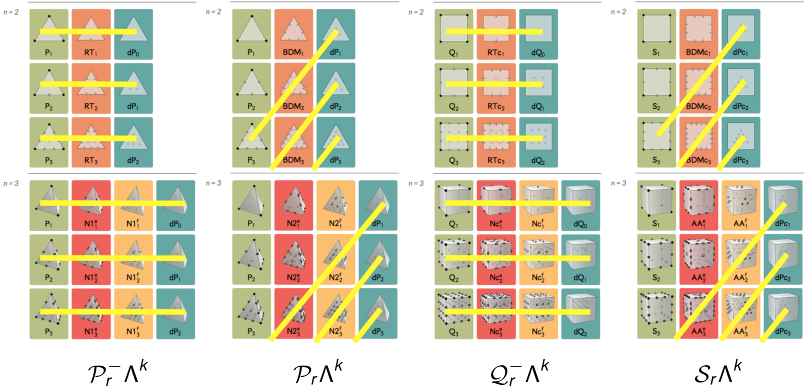
Stable pairs of elements for mixed Hodge-Laplacian problems are found by choosing consecutive spaces in compatible discretizations of the L^2 deRham Diagram.

$$H^1 \xrightarrow[\text{grad}]{\nabla} H(\text{curl}) \xrightarrow[\text{curl}]{\nabla\times} H(\text{div}) \xrightarrow[\text{div}]{\nabla\cdot} L^2$$

	H^1	$H(\text{curl})$	$H(\text{div})$	L^2
vector Poisson	σ	μ		
Maxwell's eqn's		h	b	
Darcy / Poisson			$\mathbf{u}$	p

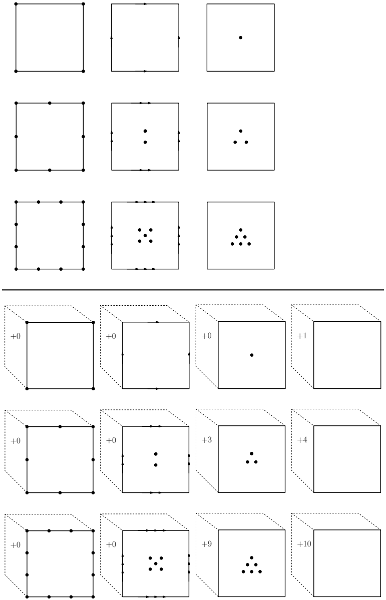
Stable pairs are found from consecutive entries in a cochain complex.

Exact cochain complexes found in the table



- Cochain complexes occur either horizontally or diagonally in the table as shown.
- Methods can be chosen from $\mathcal{P}$ or $\mathcal{P}^-$ (simplices) and $\mathcal{Q}^-$ or $\mathcal{S}$ (cubes).
- Mysteriously, the DoF count for mixed methods from the $\mathcal{P}^-$ spaces is smaller than those from the $\mathcal{P}$ spaces, while the opposite is true for $\mathcal{Q}^-$ and $\mathcal{S}$ spaces.

The 5th column: Trimmed serendipity spaces



A new column for the PToFE:
the **trimmed serendipity** elements.

$\boxed{\mathcal{S}_r^- \Lambda^k(\square_n)}$ denotes

approximation order r,
subset of k-form space $\Lambda^k(\Omega)$,
use on meshes of n-dim'l cubes.

Defined for any $n \geq 1$, $0 \leq k \leq n$, $r \geq 1$

Identical or analogous properties to all the other colummns in the table.

Computational advantage:
Fewer DoFs for mixed methods than both tensor product and serendipity counterparts.

G., Kloefkorn "Trimmed Serendipity Finite Element Differential Forms." *Mathematics of Computation*, to appear, 2019.

Mixed Method dimension comparison 1

Mixed method for Darcy problem:
$$\begin{aligned} \mathbf{u} + K\nabla p &= 0 \\ \text{div } \mathbf{u} - f &= 0 \end{aligned}$$

We compare DoF counts among the three families for use on meshes of affinely-mapped squares or cubes, when a conforming method with (at least) order r decay in the approximation of p, $\mathbf{u}$, and div $\mathbf{u}$ is desired.

Total # of degrees of freedom on a square ($n = 2$):

r	$\lvert\mathcal{Q}_r^-\Lambda^1\rvert + \lvert\mathcal{Q}_r^-\Lambda^2\rvert$	$\lvert\mathcal{S}_r\Lambda^1\rvert + \lvert\mathcal{S}_{r-1}\Lambda^2\rvert$	$\lvert\mathcal{S}_r^-\Lambda^1\rvert + \lvert\mathcal{S}_r^-\Lambda^2\rvert$
1	4+1 = 5	8+1 = 9	4+1 = 5
2	12+4 = 16	14+3 = 17	10+3 = 13
3	24+9 = 33	22+6 = 28	17+6 = 23

Total # of degrees of freedom on a cube ($n = 3$):

r	$\lvert\mathcal{Q}_r^-\Lambda^2\rvert + \lvert\mathcal{Q}_r^-\Lambda^3\rvert$	$\lvert\mathcal{S}_r\Lambda^2\rvert + \lvert\mathcal{S}_{r-1}\Lambda^3\rvert$	$\lvert\mathcal{S}_r^-\Lambda^2\rvert + \lvert\mathcal{S}_r^-\Lambda^3\rvert$
1	6+1 = 7	18+1 = 19	6+1 = 7
2	36+8 = 44	39+4 = 43	21+4 = 25
3	108+27 = 135	72+10 = 82	45+10 = 55

Mixed Method dimension comparison 2

Mixed method for Darcy problem:
$$\begin{aligned} \mathbf{u} + K\nabla p &= 0 \\ \text{div } \mathbf{u} - f &= 0 \end{aligned}$$

Interior DoFs are reduced from tensor product, to serendipity, to trimmed serendipity:

\# of **interior** degrees of freedom on a square ($n = 2$):

r	$\lvert\mathcal{Q}_r^-\Lambda_{\mathbf{0}}^1\rvert + \lvert\mathcal{Q}_r^-\Lambda_{\mathbf{0}}^2\rvert$	$\lvert\mathcal{S}_r\Lambda_{\mathbf{0}}^1\rvert + \lvert\mathcal{S}_{r-1}\Lambda_{\mathbf{0}}^2\rvert$	$\lvert\mathcal{S}_r^-\Lambda_{\mathbf{0}}^1\rvert + \lvert\mathcal{S}_r^-\Lambda_{\mathbf{0}}^2\rvert$
1	0+1 = 1	0+1 = 1	0+1 = 1
2	4+4 = 8	2+3 = 5	2+3 = 5
3	12+9 = 21	6+6 = 12	5+6 = 11

\# of **interior** degrees of freedom on a cube ($n = 3$):

r	$\lvert\mathcal{Q}_r^-\Lambda_{\mathbf{0}}^2\rvert + \lvert\mathcal{Q}_r^-\Lambda_{\mathbf{0}}^3\rvert$	$\lvert\mathcal{S}_r\Lambda_{\mathbf{0}}^2\rvert + \lvert\mathcal{S}_{r-1}\Lambda_{\mathbf{0}}^3\rvert$	$\lvert\mathcal{S}_r^-\Lambda_{\mathbf{0}}^2\rvert + \lvert\mathcal{S}_r^-\Lambda_{\mathbf{0}}^3\rvert$
1	0+1 = 1	0+1 = 1	0+1 = 1
2	12+8 = 20	3+4 = 7	3+4 = 7
3	54+27 = 81	12+10 = 22	9+10 = 19

Mixed Method dimension comparison 3

Mixed method for Darcy problem:
$$\begin{aligned} \mathbf{u} + K\nabla p &= 0 \\ \text{div }\mathbf{u} - f &= 0 \end{aligned}$$

Assuming interior DoFs could be dealt with efficiently (e.g. by static condensation), trimmed serendipity elements *still* have the fewest DoFs:

of **interface** (edge) degrees of freedom on a square ($n = 2$):

r	$\lvert\mathcal{Q}_r^-\Lambda^1(\partial\square_2)\rvert$	$\lvert\mathcal{S}_r\Lambda^1(\partial\square_2)\rvert$	$\lvert\mathcal{S}_r^-\Lambda^1(\partial\square_2)\rvert$
1	4	8	4
2	8	12	8
3	12	16	12

of **interface** (edge+face) degrees of freedom on a cube ($n = 3$):

r	$\lvert\mathcal{Q}_r^-\Lambda^2(\partial\square_3)\rvert$	$\lvert\mathcal{S}_r\Lambda^2(\partial\square_3)\rvert$	$\lvert\mathcal{S}_r^-\Lambda^2(\partial\square_3)\rvert$
1	6	18	6
2	24	36	18
3	54	60	36

Outline

Three challenges facing large-scale implementation

Given their potential, why haven't serendipity elements seen wider use?

1. Construction of "nice" basis functions is subtle
   $\rightarrow$ *Significant focus of my recent work*
2. Correct usage on unstructured quad/hex meshes is non-trivial
   $\rightarrow$ *Feasible solutions are known*
3. Need to work with established multi-purpose FEM codes
   $\rightarrow$ *Requires careful coordination with code experts*

Construction of “nice” basis functions

$\mathcal{S}_r^-\Lambda^k(\square_n)$ is a space of differential k-forms whose coefficients are polynomials in $\mathbb{R}^n$.

$$\mathcal{S}_r^-\Lambda^k = \mathcal{P}_r^-\Lambda^k \oplus \mathcal{J}_r\Lambda^k \oplus d\mathcal{J}_r\Lambda^{k-1}$$

Polynomial coefficients in each summand:

$\mathcal{P}_r^-\Lambda^k$: anything up to degree $r-1$ and some degree r

$\mathcal{J}_r\Lambda^k$: certain polynomials whose degree is between $r+1$ and $r+n-k-1$

$d\mathcal{J}_r\Lambda^{k-1}$: certain polynomials whose degree is between r and $r+n-k-2$

The “regular” serendipity space has an analogous decomposition:

$$\mathcal{S}_r\Lambda^k = \mathcal{P}_r\Lambda^k \oplus \mathcal{J}_r\Lambda^k \oplus d\mathcal{J}_{r+1}\Lambda^{k-1}$$

– – – – – – – – – – – – – –

This decomposition provides a direct sum into some precise but elaborate subspaces:

$$\mathcal{J}_r\Lambda^k(\mathbb{R}^n) := \textstyle\sum_{l\geq 1} \kappa\mathcal{H}_{r+l-1,l}\Lambda^{k+1}(\mathbb{R}^n),$$

where $\quad \mathcal{H}_{r,l}\Lambda^k(\mathbb{R}^n) := \left\{\omega \in \mathcal{H}_r\Lambda^k(\mathbb{R}^n) \mid \text{ldeg}\,\omega \geq l\right\},$

where $\quad \text{ldeg}(x^\alpha dx_\sigma) := \#\{i \in \sigma^* \ : \ \alpha_i = 1\}.$

Building a computational basis

dx	dy	dz
$-yz$	xz	0
0	$-xz$	xy
yz	xz	xy
$2xy$	x^2	0
$2xz$	0	x^2
y^2	$2xy$	0
0	$2yz$	y^2
z^2	0	$2xz$
0	z^2	$2yz$
$2xyz$	x^2z	x^2y
y^2z	$2xyz$	xy^2
yz^2	xz^2	$2xyz$

Goal: find a computational basis for $\mathcal{S}_1\Lambda^1(\square_3)$:

- Must be $H(\text{curl})$-conforming
- Must have 24 functions, 2 associated to each edge of cube
- Must recover constant and linear approx. on each edge
- The approximation space contains:

  (1) Any polynomial coefficient of at most linear order:
  $\{1, x, y, z\} \times \{dx, dy, dz\} \;\to$ 12 forms

  (2) Certain forms with quadratic or cubic order coefficients shown in table at left $\to$ 12 forms

- For constants, use "obvious" functions:

$$\{(y \pm 1)(z \pm 1)dx,\;\; (x \pm 1)(z \pm 1)dy,\;\; (x \pm 1)(y \pm 1)dz\}$$

e.g. $(y+1)(z+1)dx$ evaluates to zero on every edge ***except*** $\{y = 1, z = 1\}$ where it is $\equiv 4 \to$ constant approx.

Also, $(y+1)(z+1)dx$ can be written as a linear combo, by using the first three forms at left to get the $yz\ dx$ term

Building a computational basis

dx	dy	dz
$-yz$	xz	0
0	$-xz$	xy
yz	xz	xy
$2xy$	x^2	0
$2xz$	0	x^2
y^2	$2xy$	0
0	$2yz$	y^2
z^2	0	$2xz$
0	z^2	$2yz$
$2xyz$	x^2z	x^2y
y^2z	$2xyz$	xy^2
yz^2	xz^2	$2xyz$

- For constant approx on edges, we used:

$$\{(y\pm1)(z\pm1)dx,\ \ (x\pm1)(z\pm1)dy,\ \ (x\pm1)(y\pm1)dz\}$$

- Guess for linear approx on edges:

$$\{x(y\pm1)(z\pm1)dx,\ \ y(x\pm1)(z\pm1)dy,\ \ z(x\pm1)(y\pm1)dz\}$$

  e.g. $x(y+1)(z+1)dx$ evaluates to $4x$ on $\{y=1, z=1\}$.

- Unfortunately: $\boxed{x(y+1)(z+1)dx \notin \mathcal{S}_1\Lambda(\square_3)!}$

  Why? $x(y+1)(z+1)dx = (xyz + xy + xz + x)dx$

  but *xyz dx only appears with other cubic order coefficients!*

- Remedy: add *dy* and *dz* terms that vanish on all edges.

Building a computational basis

dx	dy	dz
$-yz$	xz	0
0	$-xz$	xy
yz	xz	xy
$2xy$	x^2	0
$2xz$	0	x^2
y^2	$2xy$	0
0	$2yz$	y^2
z^2	0	$2xz$
0	z^2	$2yz$
$2xyz$	x^2z	x^2y
y^2z	$2xyz$	xy^2
yz^2	xz^2	$2xyz$

Computational basis element associated to $\{y = 1, z = 1\}$:

$$\boxed{2x(y+1)(z+1)\ dx + (z+1)(x^2-1)\ dy + (y+1)(x^2-1)\ dz}$$

✓ Evaluates to $4x$ on $\{y = 1, z = 1\}$ (linear approx.)

✓ Evaluates to 0 on all other edges

✓ Belongs to the space $\mathcal{S}_1\Lambda(\square_3)$:

$$\begin{array}{rcrcrl}
2xyz\ dx & + & x^2z\ dy & + & x^2y\ dz & \\
2xy\ dx & + & x^2\ dy & + & 0\ dz & \\
2xz\ dx & + & 0\ dy & + & x^2\ dz & \\
2x\ dx & + & (-z-1)dy & + & (-y-1)dz & \leftarrow \text{linear order} \\
\hline
\end{array}$$

$\hookrightarrow$ summation and factoring yields the desired form)

There are 11 other such functions, one per edge. We have:

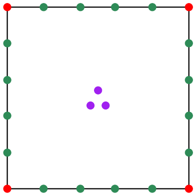
$$\begin{array}{ccccc}
\mathcal{S}_1\Lambda(\square_3) & = & \underbrace{E_0\Lambda^1(\square_3)}_{\text{“obvious” basis for constant approx}} & \oplus & \underbrace{\tilde{E}_1\Lambda^1(\square_3)}_{\text{modified basis for linear approx}} \\
\dim\ 24 & = & 12 & + & 12
\end{array}$$

A complete table of computational bases

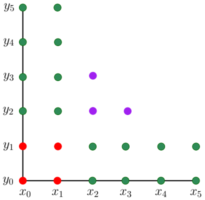
$n=3$	$k=0$	$k=1$	$k=2$	$k=3$
$\mathcal{S}_r\Lambda^k$	$V\Lambda^0(\square_3)$ $\bigoplus_{i=0}^{r-2} E_i\Lambda^0(\square_3)$ $\bigoplus_{i=4}^{r} F_i\Lambda^0(\square_3)$ $\bigoplus_{i=6}^{r} I_i\Lambda^0(\square_3)$	$\emptyset$ $\bigoplus_{i=0}^{r-1} E_i\Lambda^1(\square_3) \oplus \tilde{E}_r\Lambda^1(\square_3)$ $\bigoplus_{i=2}^{r-1} F_i\Lambda^1(\square_3) \oplus \hat{F}_r\Lambda^1(\square_3)$ $\bigoplus_{i=4}^{r} I_i\Lambda^1(\square_3)$	$\emptyset$ $\emptyset$ $\bigoplus_{i=0}^{r-1} F_i\Lambda^2(\square_3) \oplus \tilde{F}_r\Lambda^2(\square_3)$ $\bigoplus_{i=2}^{r} I_i\Lambda^2(\square_3)$	$\emptyset$ $\emptyset$ $\emptyset$ $\bigoplus_{i=2}^{r} I_i\Lambda^3(\square_3)$
$\mathcal{S}_r^-\Lambda^k$	$V\Lambda^0(\square_3)$ $\bigoplus_{i=0}^{r-2} E_i\Lambda^0(\square_3)$ $\bigoplus_{i=4}^{r} F_i\Lambda^0(\square_3)$ $\bigoplus_{i=6}^{r} I_i\Lambda^0(\square_3)$	$\emptyset$ $\bigoplus_{i=0}^{r-1} E_i\Lambda^1(\square_3)$ $\bigoplus_{i=2}^{r-1} F_i\Lambda^1(\square_3) \oplus \tilde{F}_r\Lambda^1(\square_3)$ $\bigoplus_{i=4}^{r-1} I_i\Lambda^1(\square_3) \oplus \tilde{I}_r\Lambda^1(\square_3)$	$\emptyset$ $\emptyset$ $\bigoplus_{i=0}^{r-1} F_i\Lambda^2(\square_3)$ $\bigoplus_{i=2}^{r-1} I_i\Lambda^2(\square_3) \oplus \tilde{I}_r\Lambda^2(\square_3)$	$\emptyset$ $\emptyset$ $\emptyset$ $\bigoplus_{i=2}^{r-1} I_i\Lambda^3(\square_3)$

G., Kloefkorn, Sanders "Computational serendipity and tensor product finite element differential forms."
SMAI J. Computational Mathematics, to appear, 2019.

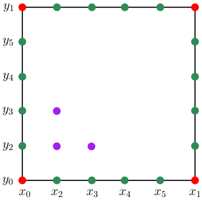
An alternate approach to basis construction



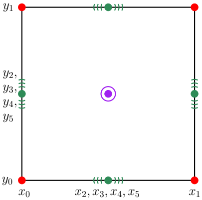
Color DoFs by type ($\mathcal{S}_5\Lambda^0$ shown)



Associate to a “lower set” of lattice points



Re-associate to the element geometry



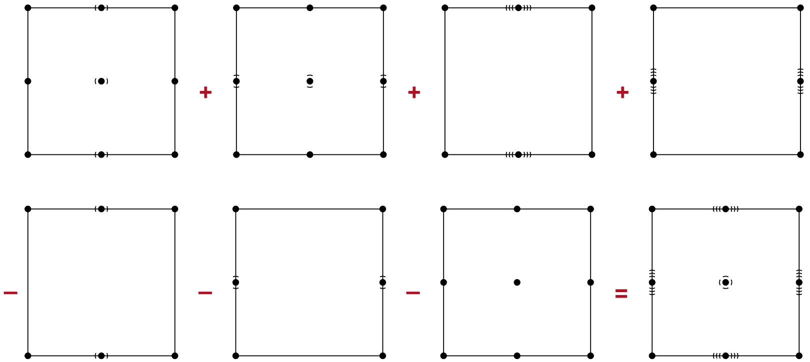
Symmetrize, and interpolate partial derivatives at edge and face midpoints

Floater, G. “Nodal Bases for the Serendipity Family of Finite Elements.” *Foundations of Computational Mathematics*, 17:4, 2017.

G., Gross, Plackowski “Numerical studies of serendipity and tensor product elements for eigenvalue problems.” *Involve*, 11:4, 2018.

An alternate approach to basis construction

This approach allows you to express serendipity basis functions **as linear combinations of tensor product** basis functions:



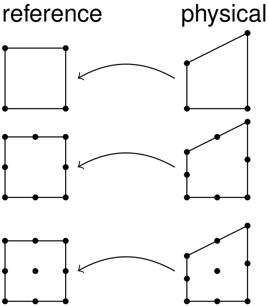
FLOATER, G. “Nodal Bases for the Serendipity Family of Finite Elements.” *Foundations of Computational Mathematics*, 17:4, 2017.

G., GROSS, PLACKOWSKI “Numerical studies of serendipity and tensor product elements for eigenvalue problems.” *Involve*, 11:4, 2018.

Correct usage on unstructured quad/hex meshes

Quadratic serendipity elements, mapped non-affinely, are only expected to converge at the rate of *linear* elements.

ARNOLD, BOFFI, FALK, "Approximation by Quadrilateral Finite Elements," *Mathematics of Computation*, 2002



	$\|\|u - u_h\|\|_{L^2}$	$\|\|\nabla(u - u_h)\|\|_{L^2}$
linear	$O(h^2)$	$O(h)$
quadratic serendipity	$O(h^2)$	$O(h)$
quadratic tensor prod.	$O(h^3)$	$O(h^2)$

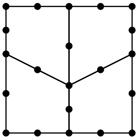
Extensions to vector-valued and higher dimensions:

ARNOLD, BOFFI, FALK, "Quadrilateral H(div) Finite Elements," *SINUM*, 2005.
ARNOLD, BOFFI, BONIZZONI, "Finite element differential forms on curvilinear cubic meshes," *Numer. Math.*, 2014

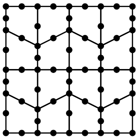
Basis functions on physical elements

Instead mapping to a reference element, use basis functions ψ_{ij} defined on physical elements:

$$u_h = I_q u := \sum_{i=1}^{n} u(\mathbf{v}_i)\psi_{ii} + u\left(\frac{\mathbf{v}_i + \mathbf{v}_{i+1}}{2}\right)\psi_{i(i+1)}$$



$n = 2$



$n = 4$

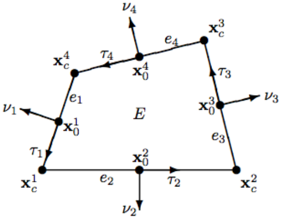
Non-affine bilinear mapping

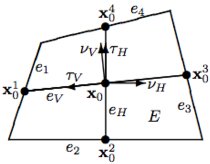
	$\|\|u - u_h\|\|_{L^2}$		$\|\|\nabla(u - u_h)\|\|_{L^2}$	
n	error	rate	error	rate
2	5.0e-2		6.2e-1	
4	6.7e-3	2.9	1.8e-1	1.8
8	9.7e-4	2.8	5.9e-2	1.6
16	1.6e-4	2.6	2.3e-2	1.4
32	3.3e-5	2.3	1.0e-2	1.2
64	7.4e-6	2.1	4.96e-3	1.1

Physical element basis functions:

	$\|\|u - u_h\|\|_{L^2}$		$\|\|\nabla(u - u_h)\|\|_{L^2}$	
n	error	rate	error	rate
2	2.34e-3		2.22e-2	
4	3.03e-4	2.95	6.10e-3	1.87
8	3.87e-5	2.97	1.59e-3	1.94
16	4.88e-6	2.99	4.04e-4	1.97
32	6.13e-7	3.00	1.02e-4	1.99
64	7.67e-8	3.00	2.56e-5	1.99

RAND, G., BAJAJ "Quadratic Serendipity Finite Elements on Polygons Using Generalized Barycentric Coordinates." *Mathematics of Computation* , 83:290, 2014.



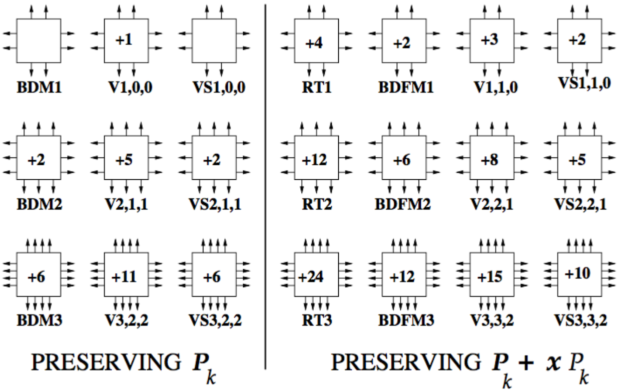


A finite element space on a general quadrilateral is built in two parts:

- Apply Piola mapping to functions associated to boundary of reference element.
- Define functions on the physical element corresponding to interior degrees of freedom in a way that ensures relevant polynomial approximation properties.

ARBOGAST, CORREA "Two families of $H(\mathrm{div})$ mixed finite elements on quadrilaterals of minimal dimension," *SIAM J. Numerical Analysis*, 2016

The virtual element technique



→ Analogues of conforming finite element spaces on squares can be treated as virtual elements.

→ Explicit basis functions are not needed to implement the method.

→ Related polygonal element methods (HHO, HDG, WG. . .) may offer similar approaches.

BEIRÃO DA VEIGA, BREZZI, MARINI, RUSSO "Serendipity face and edge VEM spaces" *Rendiconti Lincei-Matematica e Applicazioni*, 2017.

Open source finite element software packages

deal.II

→ open source C++ program library for adaptive FEM, in development since 1998

→ designed to support quad/hex meshes and *h*/*p* adaptivity

→ data structures are well-documented but not easy to introduce new element types without in-depth knowledge of the code

FEniCS Project   Firedrake

→ FEM toolkits that use Unified Form Language to define a weak form and create local assembly kernels

→ FEniCS passes kernels to DOLFIN's C++ libraries and PETSc to do solves

→ Firedrake creates intermediate data structures that are then passed to parallel schedulers, including notions like "`dofs`" and "`interior facet`" that more easily accommodate extensibility

None of these packages support (trimmed) serendipity elements yet. . .

ALNÆS ET AL. "The FEniCS Project Version 1.5" *Archive of Numerical Software*, 2015

RATHGEBER ET AL. "Firedrake: automating the finite element method by composing abstractions" *ACM Transactions on Mathematical Software*, 2016.

BANGERTH ET AL. "The `deal.ii` Library, Version 8.4," *Journal of Numerical Mathematics*, 2016

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Collaborators on this work

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Slides and Pre-prints

`http://math.arizona.edu/~agillette/`