

Research Statement

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My mathematical interests lie in the areas of geometry, probability and mathematical physics. My current research involves the application of ideas from representation theory and geometry to study families of measures on self-avoiding loops in Riemann surfaces which were introduced by Wendelin Werner. These measures are loop analogues of Schramm-Loewner evolution, and are conjectured to be a limit of (closed) self-avoiding walks on lattices.

Introduction

A *self-avoiding loop* in a surface S is a compact subset of S homeomorphic to S^1 . Werner has proven (see Theorem 2 in [13]) the existence and essential uniqueness of a family of measures $\{\mu_S : S \text{ a Riemann surface}\}$ on self-avoiding loops in Riemann surfaces satisfying the following properties:

- (i) μ_S is a positive and *non-trivial*¹ measure supported on self-avoiding loops in S .
- (ii) The family $\{\mu_S\}$ is conformally invariant in the sense that $\phi_*(\mu_S) = \mu_{S'}$ for any conformal equivalence $\phi : S \rightarrow S'$.
- (iii) If $R \subset S$ is a conformal embedding, then μ_R is equal to μ_S restricted to self-avoiding loops that stay in R .

The family $\{\mu_S\}$ is determined by the measure $\mu_0 = \mu_{\mathbb{C}}|_X$, the restriction of $\mu_{\mathbb{C}}$ to the set of self-avoiding loops

$$X = \{\gamma \subset \mathbb{C} : 0 \text{ in the bounded component of } \mathbb{C} \setminus \gamma\}. \quad (1)$$

The measure μ_0 is determined, up to an overall positive constant, by the integral formula given in Proposition 3 from [13]. The measure μ_0 is related to the chordal Schramm-Loewner evolution with parameter $\kappa = \frac{8}{3}$ and is believed to be the continuum limit for closed self-avoiding walks on a lattice.

Recent Work

I recently developed a method to compute integral formulas for μ_0 in terms of conformal welding. To put this in perspective, fix a self-avoiding loop $\gamma \in X$ and let $\phi_+ \equiv \phi_+(\gamma)$ denote the unique function which maps the unit disk \mathbb{D} conformally onto the bounded component of $\mathbb{C} \setminus \gamma$ satisfying $\phi_+(0) = 0$ and $\phi'_+(0) = \rho_0 > 0$. Likewise, we let $\phi_- \equiv \phi_-(\gamma)$ denote the unique function which maps $\mathbb{C} \setminus \overline{\mathbb{D}}$ conformally onto the unbounded component of $\mathbb{C} \setminus \gamma$ satisfying $\phi_-(\infty) = \infty$ and $\phi'_-(\infty) = \rho_\infty > 0$. The *conformal welding for γ* is the circle homeomorphism

$$\sigma \equiv \sigma(\gamma) = \phi_-^{-1} \circ \phi_+, \quad (2)$$

which is defined through an extension theorem of Carathéodory.²

¹Non-trivial in the sense that there are sets with non-zero finite measure.

²Theorem 17.5.3 in [9] implies that ϕ_\pm extend uniquely to homeomorphisms of their domain and target.

The image of μ_0 under the map $W(\gamma) = (\sigma(\gamma), \rho_\infty(\gamma))$ takes the form

$$d(W_*\mu_0) = d\nu_0(\sigma) \times \frac{d\rho_\infty}{\rho_\infty}, \quad (3)$$

where ν_0 is a finite measure (which we normalize to be a probability measure) supported on a subset of $\text{Homeo}^+(S^1)$, the orientation-preserving homeomorphisms of the circle (see Section 2 in [4]). Expectation values with respect to ν_0 can be computed. For example, Theorem 6.1 in [4] gives the simple formula

$$\mathbb{E}[u_n \bar{u}_n] = \frac{1}{n+1}, \quad (4)$$

where u_1, u_2, \dots are the coefficients in the Taylor series expansion of $u(\sigma)$, the *schlicht function*³ associated to the conformal welding $\sigma(\gamma)$.

The proof of Theorem 6.1 relies extensively on the *infinitesimal conformal invariance of μ_0 with respect to conformal vector fields* (see Section 5 in [4]):

$$\int_X (L \cdot F(\gamma)) d\mu_0(\gamma) = 0, \quad (5)$$

where F is any well-suited test function on X and \cdot indicates a certain module action of the *conformal algebra*.⁴

The module action appearing in (5) is completely natural. Any smooth real vector field V in $\mathbb{R}^2 \setminus \{0\}$ generates a one-parameter subgroup of X defined for small time. Denote this one-parameter subgroup Φ_t . If F is a function of X , then the derivative

$$V \cdot F(\gamma) := \left. \frac{d}{dt} \right|_{t=0} (F(\Phi_t \gamma)) \quad (6)$$

can be computed using a modified version of the Duren-Schiffer variational method in [7].

The conformal algebra is generated over \mathbb{C} by the real vector fields

$$V_n := L_n + \bar{L}_n \text{ and } \tilde{V}_n := i(L_n - \bar{L}_n), \quad (7)$$

where $L_n = -z^{n+1} \frac{d}{dz}$ and $\bar{L}_n = -\bar{z}^{n+1} \frac{d}{d\bar{z}}$ for $n \in \mathbb{Z}$. Section 3 in [4] gives explicit formulas for (6) corresponding to $V = V_n$ (resp. \tilde{V}_n) and $F(\gamma) = \phi_+(\gamma)$.

The *Virasoro algebra* is the lie algebra $\mathcal{Vir} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}Z$ subject to the relations

$$[Z, L_n] = [L_n, Z] = 0 \text{ and } [L_n, L_m] = (m-n)L_{n+m} + \frac{1}{12}(m^3 - m)\delta_{n+m,0}Z. \quad (8)$$

The module action we described constitutes a complex representation (denoted π_0) of a *centerless Virasoro algebra*⁵ by derivations of the polynomial algebra $A = \mathbb{C}[\rho_0, \rho_0^{-1}, u_1, u_2, \dots]$.

³A holomorphic and injective function $u : \mathbb{D} \rightarrow \mathbb{C}$ is *schlicht* if $u(0) = 0$ and $u'(0) = 1$. We adopt the convention $u(z) = z(1 + \sum_{n=1}^{\infty} u_n z^n)$.

⁴The *conformal algebra* is the Lie algebra of infinitesimal generators for conformal transformations of \mathbb{C} .

⁵The action of Z is trivial. The centerless Virasoro algebra is also called the *Witt algebra*.

I succeeded in extending the representation π_0 to include central charge $c \neq 0$, which defines a representation (denoted π_c) of the Virasoro algebra. This representation is defined by the generating function

$$\pi_c(L_n)(\phi_+) = c\phi_+(z) \left(\frac{1}{24\pi i} \oint t^{n+1} S(\phi_+^{-1}) dt \right) + \frac{z\phi'_+(z)}{4\pi i} \oint \frac{\phi_+(t)^{n+1}}{t^2\phi'_+(t)} \left(\frac{t+z}{t-z} \right) dt, \quad (9)$$

where $S(f) = \frac{f'''}{f''} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2$ is called the *Schwarzian derivative*.

The element ρ_0^{2h} generates a cyclic module called a *highest-weight representation of type* (h, c) .⁶ Different realizations of these representation are isomorphic through intertwining operators. The intertwining operator from our realization to the Kirillov-Yuriev realization constructed in [11] is given by

$$\rho_0^{2h-n} P(u_1, \dots, u_n) \mapsto P(U_1, \dots, U_n), \quad (10)$$

where $u^{-1}(z) = z(1 + \sum_{n=1}^{\infty} U_n z^n)$ and P is any n -eigenvector of the operator $\sum_{k=0}^{\infty} k u_k \frac{\partial}{\partial u_k}$.⁷ Furthermore, the representation π_c has a direct connection to the Kontsevich-Suhov deformation of Werner's family (see below). Understanding this connection is the emphasis of my current work.

Current Work

The conjectured ν_0 -distribution of the *diagonal element* $a := \frac{\rho_0}{\rho_\infty} \leq 1$ is given in [4] (see Conjecture 7.1):

$$\nu_0(\exp(-x) \leq a \leq 1) = \exp\left(-\frac{\beta_0}{x}\right) \quad (11)$$

for some $\beta_0 < \frac{5\pi^2}{4}$ and all $x > 0$. My aim is to prove this conjecture by showing the Laplace transform satisfies

$$f(x) := \int a(\sigma)^x d\nu_0(\sigma) = 2\sqrt{x\beta_0} K_1(2\sqrt{x\beta_0}), \quad (12)$$

where K_1 is a modified Bessel function.

Werner defines the function $F(r)$ as the μ_0 -measure of the set of loops in X that are contained in the annulus $\{1 < |z| < e^r\}$.⁸ The relationship between F and f is described in Section 7 of [4].

Kontsevich and Suhov conjecture⁹ the existence of a one-parameter family (parameterized by *central charge* $c \leq 1$) of locally conformally covariant measures on loops in Riemann surfaces taking values in a real oriented (*zeta-determinant*¹⁰) line bundle. If $c = 0$, then this line

⁶Highest-weight representations are the main objects of study in [8] and [10].

⁷Extensive application of the Kirillov-Yuriev representations are found for instance in [1].

⁸Cardy gives an explicit conjecture for this measure in [3].

⁹See Conjecture 1 in [12].

¹⁰If $\lambda_1 \leq \lambda_2 \leq \dots$ is the spectrum of the Laplacian Δ_g on a compact Riemannian surface (Σ, g) , then $\zeta_{\Delta_g}(s) := \sum_{j=1}^{\infty} \lambda_j^{-s}$. The ζ -regularized determinant of Δ_g is defined by $\det_{\zeta}(\Delta_g) := \exp(-\zeta'_{\Delta_g}(0))$.

bundle is trivial and the measures become scalar-valued. Therefore, the Kontsevich-Suhov family corresponding to central charge $c = \frac{(8-3\kappa)(\kappa-6)}{2\kappa} = 0$ is precisely Werner's family of measures. The Kontsevich-Suhov family can therefore be understood as a deformation of Werner's family to the values $0 < \kappa \leq 4$.

The Kontsevich-Suhov deformation is determined by a single scalar-measure μ_c on X which satisfies a strong restriction covariance property (see Definition 2.2 in [12]). The Kontsevich-Suhov family corresponding to $c = -2$ (or $\kappa = 2$) was very recently proven to exist by Benoist and Dubédat in [2]. The aim of [5] is to

- (i) Prove that μ_c (provided existence) is *infinitesimally conformally covariant*:

$$\int_X (L_n \cdot F(\gamma)) d\mu_c(\gamma) = \frac{c}{12} \int_X \mathcal{P}_{-n}(\gamma) F(\gamma) d\mu_c(\gamma), \quad (13)$$

where $z^2 S(\phi_+^{-1}) = \sum_{n=2}^{\infty} \mathcal{P}_n(\gamma) z^n$ and $\mathcal{P}_{-n}(\gamma) = \mathcal{P}_n(\rho_{\infty} b_1, \dots, \rho_{\infty}^n b_n)$ given that $\phi_-(z) = \rho_{\infty} z (1 + \sum_{n=1}^{\infty} b_n z^{-n})$.

- (ii) Use (i) to prove uniqueness of μ_c (provided existence) by adapting the proof of Theorem 6.15 from [4]. In particular, prove uniqueness of the Benoist-Dubédat SLE₂ loop measure.

Future Work

The expectation (with respect to ν_0) of *any* polynomial in $\mathbb{C}[u_1, \bar{u}_1, u_2, \bar{u}_2, \dots]$ can be calculated using the infinitesimal conformal invariance of μ_0 (see Proposition 6.13 in [4]). My aim in the near future is to compute the moment generating function for the random variable $u_n \bar{u}_n$, which can then be used to recover its distribution. This would lead to probabilistic insight to the celebrated de Brange theorem.¹¹

The diagonal distribution conjecture has a natural extension to $c \neq 0$. The image measure of μ_c under the welding map W once again takes the form

$$d(W_*\mu_c) = d\nu_c(\sigma) \times \frac{d\rho_{\infty}}{\rho_{\infty}}, \quad (14)$$

where ν_c is probability measure on a subset of $\text{Homeo}_+(S^1)$. I aim to prove the following conjecture for the ν_c -distribution of a (see Conjecture 7.10 in [4]):

$$\nu_c(\exp(-x) \leq a(\sigma) \leq 1) = \frac{\Gamma(1-c, \frac{\beta_c}{x})}{\Gamma(1-c)}, \quad (15)$$

for some constant $\beta_c > 0$ and all $x > 0$.

¹¹In [6] de Branges proved Bieberbach's famous conjecture for the size of the coefficients of schlicht functions: $|u_n| \leq n + 1$.

References

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