Homework 1

(due Friday, September 26)

If $f(x)$ is any function, then the *derivative of* $f(x)$ *at* $x = a$ is the number

$$f'(a) = \text{slope of the tangent line to } y = f(x) \text{ at } x = a,$$

which may be computed by the formula

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad (\star)$$

Example 0.1. If $f(x) = x^2$, then $f(3+h) = (3+h)^2 = 9 + 6h + h^2$ and $f(3) = 3^2 = 9$. Therefore,

$$\begin{aligned} f'(3) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{(9 + 6h + h^2) - (9)}{h} &= \lim_{h \to 0} \frac{6h + h^2}{h} \\ &= \lim_{h \to 0} (6 + h) \\ &= 6. \end{aligned}$$

Therefore, the tangent line to the graph of $y = x^2$ at $x = 3$ has slope 6. Now go to wolframalpha.com. Type

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derivative of x^2 at x=3
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into the little window. Make sure that you get 6 as a result.

1. Follow the process of Example 0.1 to compute the following derivatives using formula ($\star$). Show your work. Use Wolfram to verify your answer.

(a) $g'(2)$ if $g(x) = \frac{1}{x^2}$

(b) $P'(\sqrt{3})$ is $P(t) = \frac{2}{t} + t$

2. The following limit represents the derivative of some function f at some number a. State such an f and point a:

$$\lim_{s \to 0} \frac{\sqrt[4]{16 + s} - 2}{s}.$$

If $f(x)$ is any function, we also can think of the *derivative of* $f(x)$. Here, we are defining

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

So $f'(x)$ is now thought of as a *function of* x. The input is x and the output is the slope of the tangent line at x.

Example 0.2. If $f(x) = x^2$, then $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$ and $f(x) = x^2$. Therefore,

$$\begin{aligned} f'(3) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x^2 + 2xh + h^2) - (x^2)}{h} &= \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} (2x + h) \\ &= 2x. \end{aligned}$$

Therefore, the tangent line to the graph of $y = x^2$ at any arbitrary x-value has slope $2x$. Now go to wolframalpha.com. Type

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derivative of x^2
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into the little window. Make sure that you get $2x$ as a result.

3. Find the point (x_0, y_0) with the property that the tangent line to $y = \frac{1}{x^2}$ at x_0 is parallel to the line passing through the points $(10, 14)$ and $(12, 13)$. Give exact answers for x_0 and y_0.

4. Suppose $h(t) = -(t-3)^2 + 16$ gives the height (in cm) of an object at time t (in seconds).

(a) Compute the velocity of the ball at time t (Remember, the velocity at t is the derivative at t)

(b) Show that the exact time in which the velocity of the ball is 0 cm/s is at $t = 3$.

(c) Find the equation of the tangent line to the graph of $y = h(t)$ at the point $t = 2$.

(d) Use the tangent line approximation of $h(t)$ at $t = 2$ to approximate the ball's height at $t = 2.3$. Is this approximation an underestimate or overestimate of the true height at $t = 2.3$?

The derivative of a function $f(x)$ is very useful since it can tell us about properties of $f(x)$. We saw the following in class:

$$f'(a) > 0 \iff f(x) \text{ is increasing at } x = a$$

$$f'(a) < 0 \iff f(x) \text{ is decreasing at } x = a$$

5. Do problem 6 from section 2.3 in the book.

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6. (Algebra Practice) Solve for p exactly: $p(p+1) = 3$