Half-Plane Capacity and Conformal Radius 2

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September 23, 2014

1. Review
2. Main Proof
   - Dyadic Decomposition
   - Showing $\text{dcap}(B) \asymp |N(B)|$

Key Definitions

Suppose $A \subset \mathbb{H}$ is a bounded subset with the properties $A = \mathbb{H} \cap \overline{A}$ and $\mathbb{H} \backslash A$ simply-connected.

Definition. Let $g_A : \mathbb{H} \backslash A \longrightarrow \mathbb{H}$ with the normalization $g_A(z) = z + O(\frac{1}{z})$ as $z \to \infty$. We defined

$$\mathrm{hcap}(A) = \lim_{z \to \infty} z\big(g_A(z) - z\big)$$

so

$$g_A(z) = z + \frac{\mathrm{hcap}(A)}{z} + \cdots O(|z|^{-2})$$

as $z \to \infty$.

Key Definitions

Definition. Let Ω be simply-connected and pick a point $z_0 \in \Omega$. Define

$$\mathrm{Rad}(\Omega, z_0) := |f'(0)|,$$

where $f : \mathbb{D} \longrightarrow \Omega$ is conformal with $f(0) = z_0$. Henceforth, $\mathrm{Rad}(\Omega) = \mathrm{Rad}(\Omega, 0)$.

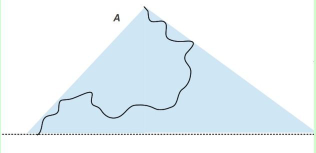
Definition. Let $B \subset \mathbb{D}$ with $0 \notin B$ and $\mathbb{D} \backslash B$ simply-connected. Define

$$\mathrm{dcap}(B) := -\log \mathrm{Rad}(\mathbb{D} \backslash B)$$

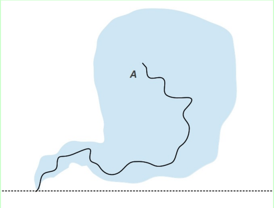
Goal: Estimate $\mathbb{H}$-Capacity

Scaling: $\text{hcap}(rA) = r^2\text{hcap}(A)$ (suggests $\text{hcap}(\cdot)$ is like an area)

Areas from Yesterday: Let $A = \gamma$ be a simple path in $\mathbb{H}$.



A Lipschitz Funtion of Norm 1



A Hyperbolic Neighborhood

Goal: Estimate $\mathbb{H}$-Capacity

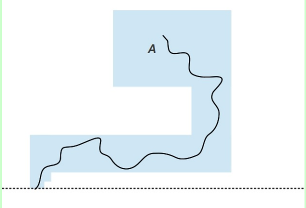
Definition. Let $\Omega \subset \mathbb{C}$ be a domain. A *Whitney square decomposition of* Ω is a collection $\mathcal{W}$ of squares $\{Q\}$ such that

(a) $\{Q\}$ are almost disjoint

(b) $\Omega = \bigcup_{Q \in \mathcal{W}} Q$

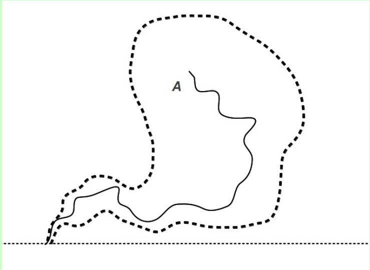
(c) $\operatorname{diam}(Q) \asymp d(Q, \partial\Omega)$

Another Area: Let $A = \gamma$ be a simple path in $\mathbb{H}$.



Whitney squares intersecting A

Theorem 1

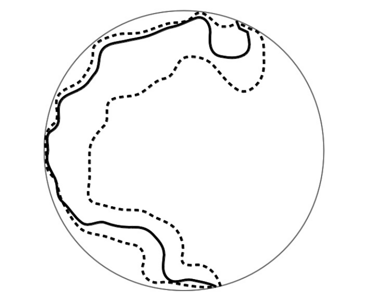


Theorem 1. Let $A \subset \mathbb{H}$ be a hull. Then

$$\text{hcap}(A) \asymp |N(A)|,$$

where $|\cdot| =$ Euclidean area. ($C_1|N(A)| \leq \text{hcap}(A) \leq C_2|N(A)|$)

Theorem 2



Theorem 2. Take $B \subset \{z \in \mathbb{D} : \frac{1}{2} < |z| < 1\}$ with $\mathbb{D} \backslash B$ simply-connected. Then

$$\mathrm{dcap}(B) \asymp |N(B)|$$

Basic Strategy

$$\text{Theorem 2} \Longrightarrow \text{Theorem 1}$$

We will now prove Theorem 2. First, we need two propositions...

Proposition 1

Fix a hull $B \subset \{z \in \mathbb{D} : \frac{1}{2} < |z| < 1\}$. For each interval $J = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$ ($n = 1, 2, \cdots$ and $k = 1, 2, \cdots, 2^n$) we define the *"dyadic square"*

$$Q_J := \{z \in \mathbb{D} : \frac{z}{|z|} \in \exp(2\pi i J) \text{ and } 1 - |z| \leq \frac{1}{2^n}\},$$

its *"top half"*

$$T(Q_j) := \{z \in Q_J : 1 - |z| > \frac{1}{2^{n+1}}\}$$

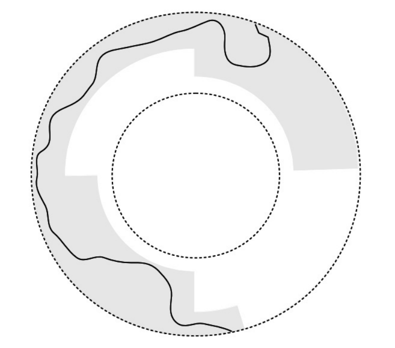
and let

$$Q(B) := \text{union of all dyadic squares } Q \text{ such that } T(Q) \cap B \neq \varnothing$$

$$(0 \in \mathbb{D} \backslash Q(B) \text{ and } \mathbb{D} \backslash Q(B) \text{ is simply-connected})$$

Proposition 1:

$$C_1|B| \leq \text{dcap}(B) \leq \text{dcap}\big(Q(B)\big) \leq C_2|Q(B)|$$



$$Q(B) = \bigcup_{T(Q)\cap B \neq \varnothing} Q$$

Proposition 1: $C_1|B| \leq \text{dcap}(B) \leq \boxed{\text{dcap}\big(Q(B)\big) \leq C_2|Q(B)|}$

The first two inequalities were shown yesterday.

Third Inequality: Write $Q(B) = \bigcup_{j=1}^{\infty} Q_j$ where $\{Q_j\}$ is a disjoint (mod ∂) family of dyadic squares arranged so that $|Q_1| \geq |Q_2| \geq \cdots$. We defined

$$K_m := f_m^{-1}(Q_m) \subset \mathbb{D},$$

where $f_m : \mathbb{D} \longrightarrow \mathbb{D} \setminus \bigcup_{j=m+1}^{\infty} Q_j$ with $f_m(0) = 0$.

We have shown

$$\text{dcap}(K_m) \asymp |Q_m| \text{ for all } m \Longrightarrow \text{dcap}\big(Q(B)\big) \leq C_2|Q(B)|.$$

Proposition 1: $C_1|B| \leq \text{dcap}(B) \leq \boxed{\text{dcap}\big(Q(B)\big) \leq C_2|Q(B)|}$

Claim: $\text{dcap}(K_m) \asymp |Q_m|$ for all m.

Idea of Proof. There exits concentric circular hulls U, V of comparable size such that

$$U \subset K_m \subset V.$$

□

Machinery

Theorem. (Beurling Projection) The probability that a Brownian motion (in a disk starting at $z = 0$) hits a closed subset E is greater than or equal to the probability that it hits its radial projection, $P(E) = \{|z| : z \in E\}$.

Machinery

Suppose $U \subset \mathbb{C}$ is a domain and $E \subset \partial U$. We fix the notation

$$\omega(z, U, E) := \mathbb{P}^z(B_{\tau_U} \in E)$$

for the harmonic measure in U from z. Here, B_t is a Brownian motion in U with $B_0 = z \in U$ and τ_U is the first exit time from U.

Example. If $U = \mathbb{D}$ and $z = 0$, then $\omega(0, \mathbb{D}, E) = \frac{1}{2\pi} \int_E dt$

Notation. We write $x \lesssim y \iff x \leq Cy$ for some constant $C > 0$.

Proposition 2: $\mathrm{dcap}(\widehat{N}(B)) \leq C\mathrm{dcap}(B)$

Let $\widehat{N}(B) := N(B) \cup \{\text{complementary components not containing } 0\}$

Define

$$N_\epsilon(B) := \text{Hyperbolic neighborhood of } B \text{ of radius } \epsilon$$

and

$$\widehat{N}_\epsilon(B) := N_\epsilon(B) \cup \{\text{complementary components not containing } 0\}$$

Claim: $\mathrm{dcap}\big(\widehat{N}_\epsilon(B)\big) \lesssim \mathrm{dcap}(B)$

Claim. $\mathrm{dcap}\big(\widehat{N}_\epsilon(B)\big) \lesssim \mathrm{dcap}(B)$ for $\epsilon > 0$.

If we prove the claim, then

$$\mathrm{dcap}\Big(\widehat{N}_{\frac{1}{\epsilon}}\big(\widehat{N}_\epsilon(B)\big)\Big) \lesssim \mathrm{dcap}\big(\widehat{N}_\epsilon(B)\big) \quad (\star)$$

$\mathrm{dcap}\big(\widehat{N}(B)\big) \lesssim$ LHS of $(\star)$ follows from yesterday's lemma:

Lemma. Given hulls $\Omega \subseteq \Omega' \subset \mathbb{D}$ $(0 \notin \Omega)$. Then $\mathrm{dcap}(\Omega) \leq \mathrm{dcap}(\Omega')$

So **Claim $\Longrightarrow$ Proposition 2**

Claim: $\mathrm{dcap}\big(\widehat{N}_\epsilon(B)\big) \lesssim \mathrm{dcap}(B)$

For $n \geq 1$ define

$$D_n := \{z \in \mathbb{D} : \frac{1}{2^{n+1}} \leq 1 - |z| < \frac{1}{2^n}\} \quad \text{and} \quad B_n := B \cap D_n$$

If $a \in D_n$ and $b \in D_{n+2}$, then $2^{n-1} d(a,b) < d_h(a,b) \leq 2^{n+3} d(a,b)$ and

$$d_h(D_n, D_{n+2}) \asymp 1.$$

Choose $\epsilon > 0$ so that for all n,

$$\mathcal{B}(z, 2\epsilon) \text{ is contained in } D_{n-1} \cup D_n \cup D_{n+1}$$

Claim: dcap$\left(\widehat{B}\right) \lesssim$ dcap(B)

Proof of Claim. Let $\omega_n(z) := \omega(z, \mathbb{D}\backslash B, B_n)$ denote the measure of B_n. Let $f : \mathbb{D} \longrightarrow \mathbb{D}\backslash B$ with $f(0) = 0$ and $f'(0) > 0$. The mean value property for $z \mapsto \log\left|\frac{f}{z}\right|$ lets us write

$$\log|f'(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{it})|\, dt = \int_{\partial\mathbb{D}} \log|f(w)|\, \omega(0, \mathbb{D}, dw) \quad (\star)$$

The fact $1 - x \asymp -\log x$ for $\frac{1}{2} \leq x < 1$ along with $(\star)$ implies

$$\text{dcap}(B) \asymp \sum_{n=1}^{\infty} \int_{B_n} (1 - |z|)\, \omega(0, \mathbb{D}\backslash B, dz) \asymp \sum_{n=1}^{\infty} \frac{\omega_n(0)}{2^n}.$$

Claim: $\text{dcap}(\widehat{B}) \lesssim \text{dcap}(B)$

We have just shown that

$$\text{dcap}(B) \asymp \sum_{n=1}^{\infty} \frac{\omega_n(0)}{2^n}$$

Additionally, we define

$$\widehat{\omega}_n(z) = \omega(z, \mathbb{D} \backslash \widehat{N}_\epsilon(B), D_n \cap \widehat{N}_\epsilon(B))$$

Following the same ideas, we have

$$\text{dcap}(\widehat{B}) \asymp \sum_{n=1}^{\infty} \frac{\widehat{\omega}_n(0)}{2^n},$$

where $\widehat{N}_\epsilon(B) = \widehat{B}$.

Claim: $\text{dcap}(\widehat{B}) \lesssim \text{dcap}(B)$

We show

$$\widehat{\omega}_n(z) \lesssim \omega_{n-1}(z) + \omega_n(z) + \omega_{n+1}(z) \quad (\star)$$

for all n and $z \in \mathbb{D} \backslash \widehat{B}$:

Harmonic in $\mathbb{D} \backslash \widehat{B}$ + max. principle $\Longrightarrow$ take $z \in \partial\mathbb{D} \backslash \widehat{B}$.

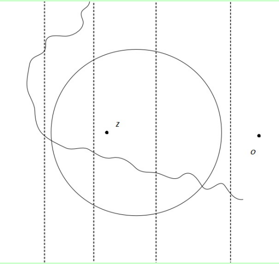
If $z \notin D_n \Longrightarrow \widehat{\omega}_n(z) = 0$. We assume $z \in D_n$. Since $\widehat{\omega}_n(z) \leq 1$, we just need to show

$$\omega_{n-1}(z) + \omega_n(z) + \omega_{n+1}(z) = \omega(z, B_{n-1} \cup B_n \cup B_{n+1}, \mathbb{D} \backslash B)$$

is bounded away from 0. This follows from the Beurling Projection theorem. So $(\star)$ holds.

Claim: $\mathrm{dcap}(\widehat{B}) \lesssim \mathrm{dcap}(B)$

We have shown $\widehat{\omega}_n(0) \lesssim \omega_{n-1}(0) + \omega_n(0) + \omega_{n+1}(0)$ for all n.



Divide by 2^n and form a sum:

$$\sum_{n=1}^{\infty} 2^{-n}\widehat{\omega}_n(0) \asymp \mathrm{dcap}(\widehat{N}_\epsilon(B)) \lesssim \frac{7}{2}\sum_{n=1}^{\infty} 2^{-n}\omega_n(0) = \frac{7}{2}\mathrm{dcap}(B).$$

□

$\mathrm{dcap}(B) \asymp |N(B)|$

$$|B| \lesssim \mathrm{dcap}(B) \overset{\textbf{Prop. 1}}{\leq} \mathrm{dcap}\big(Q(B)\big) \lesssim |Q(B)| \qquad \mathrm{dcap}\big(\widehat{N}(B)\big) \overset{\textbf{Prop. 2}}{\lesssim} \mathrm{dcap}(B)$$

Proof of Theorem 2:

- $\mathrm{dcap}(B) \underset{\text{Prop. 2}}{\gtrsim} \mathrm{dcap}\big(\widehat{N}(B)\big) \underset{\text{Prop. 1}}{\gtrsim} |\widehat{N}(B)| \geq |N(B)|$

- $\mathrm{dcap}(B) \underset{\text{Prop. 1}}{\leq} \mathrm{dcap}\big(Q(B)\big) \leq \mathrm{dcap}\big(Q(\widehat{N}(B)\big) \underset{\text{Prop. 1}}{\lesssim} |Q\big(\widehat{N}(B)\big)| \lesssim |N(B)|$

□

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