Homework 6: Convergence vs. Divergence (due February 23)

1. Use the behavior of "rational" functions (for large values of x) to decide whether the following integrals converge or diverge. After you make your claim, then do a rigorous arguement to show that they converge/diverge (by comparing the integral to a convergent/divergent integral through a series of inequalities). The first one is worked out for you as an example of what I am looking for. Remember, this only works if we are dealing with integrals with $\pm\infty$ as bounds of integration.

(a) $\displaystyle\int_3^\infty \frac{x^2-1}{x^3-5}\,dx$

Solution. For large values of x, $x^2-1 \approx x^2$ and $x^3-5 \approx x^3$. Therefore, $\dfrac{x^2-1}{x^3-5} \approx \dfrac{x^2}{x^3} = \dfrac{1}{x}$ as x gets very large. Since the integral $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges ($p=1$), we suspect $\displaystyle\int_3^\infty \frac{x^2-1}{x^3-5}\,dx$ should also **DIVERGE.**

To show that this is infact true, we must use a chain of inequalities to show $\dfrac{x^2-1}{x^3-5}$ is greater than or equal to a function with a divergent integral. To this end, we observe

$$\frac{x^2-1}{x^3-5} \geq \frac{(0.5)x^2}{x^3-5} \geq \frac{(0.5)x^2}{x^3} = \frac{1}{2x}.$$

Since $\displaystyle\int_3^\infty \frac{1}{2x}\,dx$ diverges ($p=1$), we must have that $\displaystyle\int_3^\infty \frac{x^2-1}{x^3-5}\,dx$ also diverges (using the "comparison test"). ✓

(b) $\displaystyle\int_1^\infty \frac{x^2-1}{x^4+3}\,dx$

(c) $\displaystyle\int_2^\infty \frac{y^4+\sqrt{y}}{5y^7+10}\,dy$

(d) $\displaystyle\int_4^\infty \frac{t^3+t-1}{t^4+t+1}\,dt$

2. Show that $\displaystyle\int_0^1 \frac{1}{\sqrt{\theta^3+\theta}}\,d\theta$ converges

3. Show that $\displaystyle\int_0^\pi \frac{2-\sin(r)}{r^2}\,dr$ diverges