Half-Plane Capacity and Conformal Radius 1

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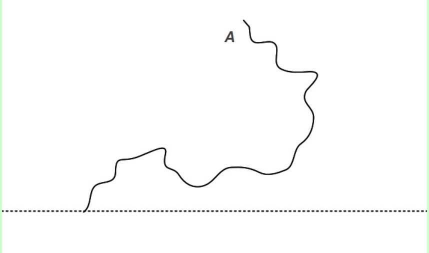
The Half-Plane Capacity

Let $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ denote the half-plane.

Definition. A bounded subset $A \subset \mathbb{H}$ is called a *hull* if $A = \overline{A} \cap \mathbb{H}$ and $\mathbb{H} \backslash A$ is simply-connected.

Example. If $A = \gamma$ is a simple path in $\mathbb{H}$ starting on $\partial\mathbb{H}$:



The Half-Plane Capacity

Lemma. If $A \subset \mathbb{H}$ is a hull, then there exits a unique conformal map $g_A : \mathbb{H} \backslash A \longrightarrow \mathbb{H}$ satisfying

$$g_A(z) = z + O(\frac{1}{z})$$

as $z \to \infty$.

Idea of Proof. Riemann's mapping theorem guarantees that there exists a conformal map $g : \mathbb{H} \backslash A \longrightarrow \mathbb{H}$ with $|g(z)| \to \infty$ as $z \to \infty$. A is contained in some ball of finite radius (centered at 0). The Schwarz reflection principle $\Longrightarrow$ g can be extended to a conformal map on $\mathbb{C} \backslash \{\text{this ball}\}$. Compose with conformal map $\mathbb{H} \to \mathbb{H}$.

□

The Half-Plane Capacity

Definition. Let $A \subset \mathbb{H}$ be a hull and $g_A : \mathbb{H} \backslash A \longrightarrow \mathbb{H}$ be the unique conformal map satisfying $g_A(z) = z + O(\frac{1}{z})$ as $z \to \infty$.

The *half-plane capacity of* A is defined as

$$\text{hcap}(A) := \lim_{z \to \infty} z\left(g_A(z) - z\right).$$

i.e.,

$$g_A(z) = z + \frac{\text{hcap}(A)}{z} + \cdots$$

The Half-Plane Capacity

Example (from [2]). Take $A = \overline{\mathbb{D}} \cap \mathbb{H}$ with $g_A(z) = z + \frac{1}{z}$.

Half-circles Re^{it} ($R \geq 1$ and $0 < t < \pi$) map to

$$\left((R + \frac{1}{R}) \cos t, (R - \frac{1}{R}) \sin t\right).$$

The line segments from ± 1 to $\pm\infty$ along real axis map to line segments from ± 2 to $\pm\infty$. The boundary arc maps to line segment connecting ± 2.

$\text{hcap}(A) = 1$.

The Half-Plane Capacity

hcap$(\cdot)$ **is monotone:** If $A \subseteq A'$, then $g_{A'} = g_\Omega \circ g_A$, where $\Omega = g_A(A' \backslash A)$. Therefore, $\text{hcap}(A') = \text{hcap}(A) + \text{hcap}\big(g_A(A' \backslash A)\big)$

Lemma. Monotonicity of hcap$(\cdot)$ is strict.

Define $b(t) := \text{hcap}\big(\gamma(0,t]\big)$ for a simple curve $\gamma : [0,\infty) \longrightarrow \mathbb{C}$ with $\gamma(0) \in \mathbb{R}$ and $\gamma(0,\infty) \subset \mathbb{H}$.

Lemma. b is continuous and strictly increasing.

γ may be reparameterized by hcap: $\widetilde{\gamma}(t) = \gamma\big(b^{-1}(2t)\big)$ and

$$\text{hcap}\big(\widetilde{\gamma}(0,t]\big) = 2t$$

Disk Capacity

Definition. Let Ω be simply-connected. The *conformal radius of* Ω *with respect to* $z \in \Omega$ is the positive number

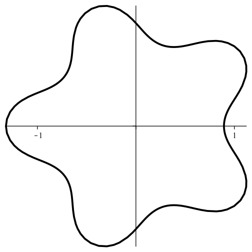
$$\mathrm{Rad}(\Omega, z) := |f'(0)|,$$

where $f : \mathbb{D} \longrightarrow \Omega$ is conformal with $f(0) = z$.

Example. $\mathrm{Rad}(\mathbb{D}(r), 0) = r$

Disk Capacity

Example. Take Ω to be the inside of the *amoeba*:



Then $f(z) = \frac{z}{(1+\frac{3}{4}z^5)^{\frac{1}{5}}}$ gives a uniformization $f : \mathbb{D} \longrightarrow \Omega$. And we have $\mathrm{Rad}(\Omega, 0) = 1$.

Disk Capacity

Lemma. If $\Omega \subseteq \Omega'$ are simply-connected with $0 \in \Omega$, then

$$\mathrm{Rad}(\Omega, 0) \leq \mathrm{Rad}(\Omega', 0)$$

Proof. Let $f : \mathbb{D} \longrightarrow \Omega$ and $g : \mathbb{D} \longrightarrow \Omega'$ with $f(0) = g(0) = 0$. The map

$$\phi : \mathbb{D} \longrightarrow \mathbb{D}\,; \quad \phi := g^{-1} \circ f$$

is holomorphic with $\phi(0) = 0$. By Schwarz's lemma, we have $|\phi'(0)| \leq 1 \iff |(g^{-1})'(0) f'(0)| \leq 1 \iff |g'(0)| \leq |f'(0)|$. □

Disk Capacity

Example. Suppose B is a hull in $\mathbb{D}$ and

$$\phi : \mathbb{D} \backslash B \longrightarrow \mathbb{D}$$

with $\phi(0) = 0$ and $\phi'(0) > 1$. Let γ be a radial $\text{SLE}_{8/3}$ from 1 to 0 in $\mathbb{D}$. Then

$$\mathbb{P}(\gamma(0, \infty) \cap B \neq \varnothing) = \text{Rad}(\mathbb{D} \backslash B, 0)^{-\frac{5}{48}} |\text{Rad}(\mathbb{D} \backslash B, \phi^{-1}(1)|^{-\frac{5}{8}}$$

Disk Capacity

Definition. A hull in $\mathbb{D}$ is the same as a hull in $\mathbb{H}$: A $B \subset \mathbb{D}$ with $0 \notin B$ and $\mathbb{D} \backslash B$ simply-connected. The *disk capacity of* B is the positive number

$$\mathrm{dcap}(B) := -\log \mathrm{Rad}(\mathbb{D} \backslash B, 0)$$

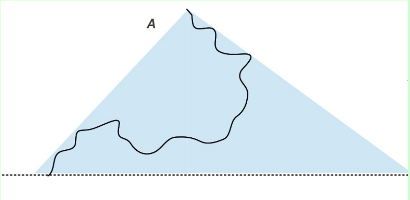
$\left(\text{Our preview lemma} \implies \mathrm{Rad}(\mathbb{D} \backslash B, 0) < 1\right)$

$\mathrm{dcap}(\cdot)$ **is relevant:** Plays the role of hcap in radial SLE

Geometry

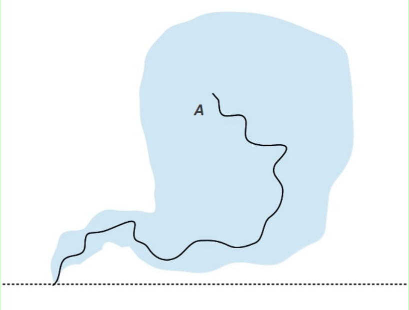
Goal: Estimate hcap(A) in terms of geometric (simpler) quantities associated to A.

Example. A Lipschitz function of norm 1 lying above $A = \gamma$:



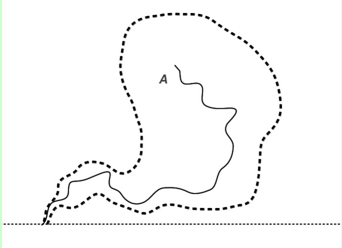
Geometry

Example. Hyperbolic distance in $\mathbb{H}$: $d_h(a,b) = \inf_\sigma \left(\int_\sigma \frac{|dz|}{\mathrm{Im}(z)} \right)$



We can form a *hyperbolic neighborhood* of $A = \gamma$

Theorem 1



Theorem 1 (Theorem 1.1 in [2]). Let A be a hull in $\mathbb{H}$. Then

$$\mathrm{hcap}(A) \asymp |N(A)|,$$

where $|\cdot| =$ Euclidean area, i.e., there exist absolute constants $C_1, C_2 > 0$ such that $C_1|N(a)| \leq \mathrm{hcap}(A) \leq C_2|N(a)|$

Theorem 2

Theorem 2 (Theorem 1.2 in [2]). If $B \subset \{z \in \mathbb{D} : \frac{1}{2} < |z| < 1\}$ and $\mathbb{D} \backslash B$ is simply-connected, then

$$\mathrm{dcap}(B) \asymp |N(B)|$$

Connection Between Theorems 1 and 2

- Tomorrow we show Theorem 2 $\Longrightarrow$ Theorem 1
- Today we begin with the proof of Theorem 2

Setup

Before proving our first Proposition we state a useful fact.

Lemma: Suppose $f(z) = \sum_{k=1}^{\infty} a_k z^k$ is holomorphic and injective on $\mathbb{D}$ with $f(0) = 0$ and $f'(0) > 0$. Then

$$|\text{Image of } f| = \pi \sum_{k=1}^{\infty} k|a_k|^2$$

Proposition 1

Fix $B \subset \{z \in \mathbb{D} : \frac{1}{2} < |z| < 1\}$ with $\mathbb{D} \backslash B$ simply-connected...

Consider the hyberbolic distance d_h on $\mathbb{D}$:

$$d_h(a, b) = \inf_\gamma \left(\int_\gamma \frac{|dz|}{1-|z|^2} \right)$$

If $|z| = \frac{1}{2}$, then $d_h(0, z) = \log 3 > 1$. Therefore,

$$0 \notin N(B) = \text{hyperbolic neighborhood of radius 1}$$

Proposition 1

Let $J = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$ for $n = 1, 2, \cdots$ and $k = 1, 2, \cdots, 2^n$. Define the *"dyadic square"*

$$Q_J := \{z \in \mathbb{D} \, : \, \frac{z}{|z|} \in \exp(2\pi i J) \text{ and } 1 - |z| \leq \frac{1}{2^n}\},$$

its *"top half"*

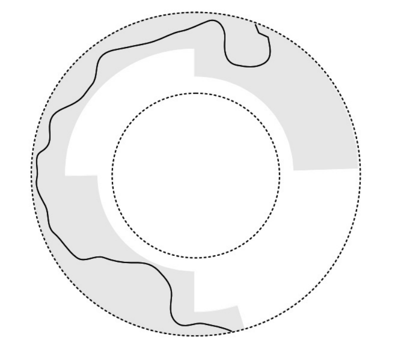
$$T(Q_j) := \{z \in Q_J \, : \, 1 - |z| > \frac{1}{2^{n+1}}\}$$

and let

$$Q(B) := \text{union of all dyadic squares } Q \text{ such that } T(Q) \cap B \neq \varnothing$$

$$(0 \in \mathbb{D} \backslash Q(B) \text{ and } \mathbb{D} \backslash Q(B) \text{ is simply-connected})$$

Proposition 1: $C_1|B| \leq \text{dcap}(B) \leq \text{dcap}\big(Q(B)\big) \leq C_2|Q(B)|$



$$Q(B) = \bigcup_{T(Q)\cap B\neq\varnothing} Q$$

Proposition 1: $\boxed{C_1|B| \leq \text{dcap}(B)} \leq \text{dcap}(Q(B)) \leq C_2|Q(B)|$

Let $f : \mathbb{D} \longrightarrow \mathbb{D}\backslash B$ with $f(0) = 0$ and $f'(0) > 0$. We will write

$$f(z) = a_1 z + a_2 z^2 + \cdots = \sum_{k=1}^{\infty} a_k z^k \quad (\,\text{Rad}(\mathbb{D}\backslash B) = a_1\,)$$

Lemma $\Longrightarrow |\mathbb{D}\backslash B| = \pi \sum_{k=1}^{\infty} k|a_k|^2 = \pi - |B|$. Therefore,

$$\frac{|B|}{4\pi} = \frac{1}{4}(1 - \sum_{k=1}^{\infty} k|a_k|^2) \leq \frac{1}{4}(1 - a_1^2) \leq -\log(a_1).$$

□

Proposition 1: $C_1|B| \leq \boxed{\mathrm{dcap}(B) \leq \mathrm{dcap}\big(Q(B)\big)} \leq C_2|Q(B)|$

Let $\Omega \subseteq \Omega' \subseteq \mathbb{D}$ be simply-connected regions with $0 \in \Omega$. We have shown $\mathrm{Rad}(\Omega, 0) \leq \mathrm{Rad}(\Omega', 0)$:

Let $f : \mathbb{D} \longrightarrow \Omega$ and $g : \mathbb{D} \longrightarrow \Omega'$ with $f(0) = g(0) = 0$. The map

$$\phi : \mathbb{D} \longrightarrow \mathbb{D}\,; \quad \phi := g^{-1} \circ f$$

is holomorphic with $\phi(0) = 0$. By Schwarz's lemma, we have $|\phi'(0)| \leq 1 \iff |(g^{-1})'(0) f'(0)| \leq 1 \iff |g'(0)| \leq |f'(0)|$.

Therefore,

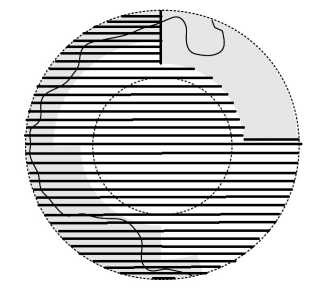
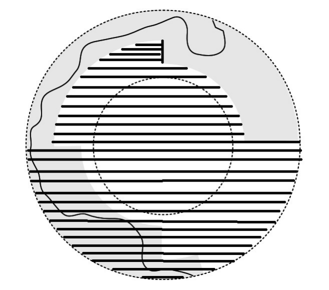
$$-\log\Big(\mathrm{Rad}(\Omega)\Big) \leq -\log\Big(\mathrm{Rad}(\Omega')\Big)$$

$\square$

Proposition 1: $C_1|B| \le \mathrm{dcap}(B) \le \boxed{\mathrm{dcap}\big(Q(B)\big) \le C_2|Q(B)|}$

Write $Q(B) = \bigcup_{j=1}^{\infty} Q_j$ where $\{Q_j\}$ is a disjoint (mod ∂) family of dyadic squares arranged so that $|Q_1| \ge |Q_2| \ge \cdots$.

If we show $\mathrm{dcap}\big(\bigcup_{j=m}^{\infty} Q_j\big) - \mathrm{dcap}\big(\bigcup_{j=m+1}^{\infty} Q_j\big) \asymp |Q_m|$ for all m, then the claim follows.

Proposition 1: $C_1|B| \le \mathrm{dcap}(B) \le \boxed{\mathrm{dcap}\big(Q(B)\big) \le C_2\big|Q(B)\big|}$

To show $\mathrm{dcap}\big(\bigcup_{j=m}^{\infty} Q_j\big) - \mathrm{dcap}\big(\bigcup_{j=m+1}^{\infty} Q_j\big) \asymp |Q_m|$ let

$$R_m = \mathrm{Rad}(\mathbb{D} \setminus \bigcup_{j=m}^{\infty} Q_j).$$

Then

$$|Q(B)| \asymp -\sum_{m=1}^{\infty} \log\left(\frac{R_m}{R_{m+1}}\right) = -\log\left(\prod_{m=1}^{\infty} \frac{R_m}{R_{m+1}}\right) = -\log \mathrm{Rad}\big(\mathbb{D} \setminus Q(B)\big)$$

Proposition 1: $C_1|B| \leq \mathrm{dcap}(B) \leq \boxed{\mathrm{dcap}\big(Q(B)\big) \leq C_2|Q(B)|}$

Now we define

$$K_m := f_m^{-1}(Q_m) \subset \mathbb{D},$$

where $f_m : \mathbb{D} \longrightarrow \mathbb{D} \setminus \bigcup_{j=m+1}^{\infty} Q_j$ with $f_m(0) = 0$. Observe

$$\mathrm{dcap}\Big(\bigcup_{j=m}^{\infty} Q_j\Big) - \mathrm{dcap}\Big(\bigcup_{j=m+1}^{\infty} Q_j\Big) = -\log\left(\frac{f'_{m-1}(0)}{f'_m(0)}\right) = -\log\Big((\phi_m)'(0)\Big),$$

where $\phi_m := f_m^{-1} \circ f_{m-1} : \mathbb{D} \longrightarrow \mathbb{D} \setminus K_m$. Therefore,

$$\mathrm{dcap}\Big(\bigcup_{j=m}^{\infty} Q_j\Big) - \mathrm{dcap}\Big(\bigcup_{j=m+1}^{\infty} Q_j\Big) = \mathrm{dcap}(K_m).$$

Proposition 1: $C_1|B| \leq \text{dcap}(B) \leq \boxed{\text{dcap}\big(Q(B)\big) \leq C_2|Q(B)|}$

We have shown

$$\text{dcap}(K_m) \asymp |Q_m| \text{ for all } m \Longrightarrow \text{dcap}(Q(B)) \leq C_2|Q(B)|.$$

Claim: $\text{dcap}(K_m) \asymp |Q_m|$ for all m.

Sketch of Proof. There exits concentric circular hulls U, V such that

$$U \subset K_m \subset V$$

□

Proposition 1: $C_1|B| \leq \text{dcap}(B) \leq \boxed{\text{dcap}\big(Q(B)\big) \leq C_2|Q(B)|}$

To Be Continued...

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